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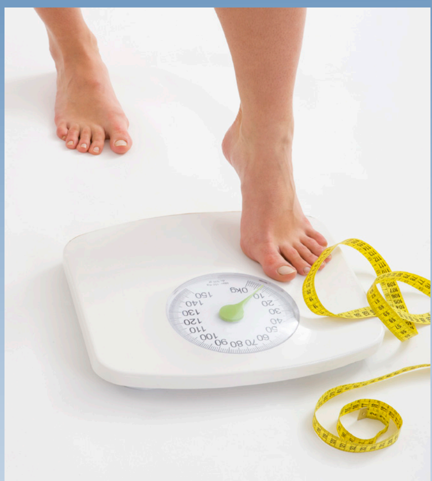

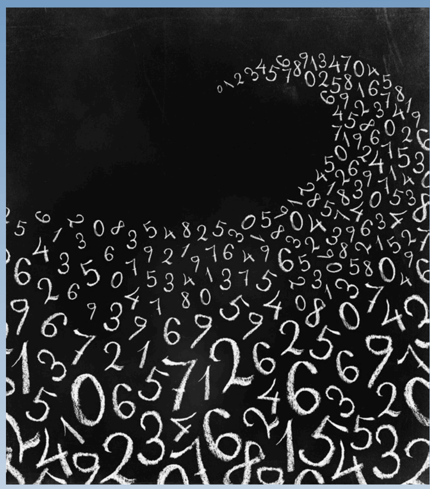

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A Fuzzy Mathematical Model of College Freshman Weight Gain

Chelsea Fischer
Department of Exercise Science
Creighton University
Omaha, Nebraska 68178
email: ChelseaFischer@creighton.edu

Abstract:

We use fuzzy mathematics to develop a system for measuring how well a United States college or university is helping to prevent weight gain and increase awareness of health among its incoming freshman. The methodology includes the use of linear equations that place a specific weight on each of the five components that make up the overarching goal. Expert opinion is the basis for the weights that will be assigned to each component. Then we will ascertain which goals are most essential to prevent the problem and determine the level of agreement within the group of experts.

Keywords:

Obesity; college freshmen; Guiasu method; analytic hierarchy process; Dempster-Shafer; theory; fuzzy preference relations; core; consensus winner

1. Introduction

As obesity becomes an epidemic problem across the United States the impact on health and care costs is causing major health concerns. Some of the health conditions that are associated with obesity include cancer, type 2 diabetes, heart disease, hypertension, stroke, sleep apnea and dyslipidemia, [13]. While this trend is occurring across all age groups, it is affecting the adult population to the greatest extent. Sometime between the innocence of youth and the adult working life a number of changes take place that contribute to weight gain. Most adults who are obese become so as a result of a gradual weight gain initiating earl in adulthood, [12]. The transition from high school to college is well known for being a time of weight gain in a large portion of the population, [2]. This is of particular importance for colleges and universities at which freshman live on campus and consume food with in dining halls.

As a result, one area that colleges and universities need to focus on in their freshman orientation, throughout the first year of education and in their core curriculum is maintenance of a healthy life-style including prevention of weight gain. This prevention of weight gain among freshman is the overarching goal and there are various goals that are needed to be addressed in order to reach this overarching goal that impacts the weight of early college students.

$G1$: The food consumed by the freshman students assuming an on campus living arrangement, [1].

$G2$: The level of physical activity and exercise that a student performs daily in order to maintain or improve fitness levels, [4].

$G3$: The time constraints put on a college student that forces a level of prioritization to occur through out the student's life, [12].

$G4$: The level of stress a college student is under has large impact on weight gain, [14].

$G5$: The social influences that are constantly impacting a college student can lead to weight gain, [12].

We present methods for measuring the success a college or university is having in achieving its overarching goal of prevention of weight gain in college freshmen. These methods are developed using three approaches: the Guiasu method [3], the analytic hierarchy process (AHP) [10, 11], and a method based on Dempster-Shafer theory [6]. The three approaches yield linear equations with $G1$-$G5$ as the independent variables and the overarching goal as the dependent variable. All three approaches rely on expert opinion. The resulting models weights the relative importance of the five subgoals $G1$-$G5$ of the overarching goal. We asked experts to score on a scale from 0 to 10 how well the university is achieving the five subgoals. These numbers were substituted into the linear equations to yield a number between 0 and 10 that measured the extent to which the overarching goal was being achieved.

Within each of the subgoals, $G1$-$G5$, there are a number of factors that have a large impact on each of the subgoals and need to be addressed by each college or university being considered in order to be able to help steer a student in the right direction. These factors are listed in Appendix A. In Appendix B, we determine linear equations for each subgoal in terms of its factors. These factors are expressed in such a way that it is understood which goal they contribute to. Within each of the goals a linear equation in terms of the factors will be given based on expert opinion as the factors importance within the goal. Although the work is based to a degree on meetings with experts in various health fields this does not reflect an official position or an endorsement by any of these groups.

2. Models

In this section, we present the linear equations that are used to measured how well a university is doing in achieving its goal of controlling weight gain in freshmen. The following table gives the weights the experts $E1, E2, E3, E4$ assigned to the subgoals $G1$-$G5$ as to their importance with respect to the overarching goal.

	$E1$	$E2$	$E3$	$E4$	Row Avg.
$G1$	10	10	10	9	$9.75 = w_1$
$G2$	10	10	10	9	$9.75 = w_2$
$G3$	1	8	4	8	$5.25 = w_3$
$G4$	5	2	2	7	$4.00 = w_4$
$G5$	8	5	5	7	$6.25 = w_5$
Column Total	34	35	31	40	35.00

In the following table, we normalize the columns of the previous table.

Cols. Normalized	$E1$	$E2$	$E3$	$E4$	Row Avg	Row Product
$G1$	.29	.29	.32	.22	.28	.0059206
$G2$	.29	.29	.32	.22	.28	.0059206
$G3$	.03	.23	.13	.20	.15	.0001794
$G4$	.15	.06	.06	.17	.11	.0000918
$G5$	.24	.14	.16	.17	.18	.0009139

It is shown in [7] that the row averages from the second table represent the weights of the Gi as determined by the Guiasu method. This occurs because of the probabilistic nature of the probability assignments. Thus

$$G = .28G1 + .28G2 + .15G3 + .11G4 + .18G5.$$

Next the analytic hierarchy process will be considered. We take the row averages of the first matrix to determine the 5×5 reciprocal matrix R of the analytic hierarchy process, where the ij-th entry in R is w_i/w_j. We then normalize the columns of R to obtain a matrix we call N. It is known that each column of N will be the same, [10, 11]. Hence it suffices to normalize the first column of R. we have that $\sum_{i=1}^{5} w_i = 35$. Hence the $j1$-th entry of N is $(w_j/w_1)/(35.w_1) = w_j/35$, for $j = 1, ..., 5$. Hence the first column and consequently all the columns of N are the transpose of the vector $(.28, .28, .15, .11, .18)$. Thus

$$G = .28G1 + .28G2 + .15G3 + .11G4 + .18G5.$$

We notice that the equations for the Guiasu method and the analytic hierarchy process are the same after rounding. In [9], we give a necessary and sufficient condition for them to be identical.

We next consider the Dempster-Shafer method. We normalize the row product column by dividing each entry by the sum of the column entries which is .0130263. We obtain the transpose of the vector $(.45, .45, .01, .01, .07)$. Thus

$$G = .45G1 + .45G2 + .01G3 + .01G4 + .07G5.$$

The theoretical underpinnings of this method can be found in [9].

3. Fuzzy Preference Relations

We use techniques from [5] to determine the degree to which the experts considered each goal stronger than the others. Let $X = \{G1, G2, G3, G4, G5, G6\}$. We use the following fuzzy binary relations $\rho_k, k = 1, ..., 4$, on X to give the four experts' preferences. Define $\rho_k : X \times X \to [0, 1]$ by $\forall (Gi, Gj) \in X \times X$,

$$\rho_k(Gi, Gj) = \begin{cases} (Gi - Gj + .5) \wedge 1 \text{ if } Gi \geq Gj, \\ 1 - [(Gj - Gi + .5) \wedge 1] \text{ if } Gi < Gj, \end{cases}$$

where $k = 1, ..., 4, i, j = 1, ..., 6$. Let $r_{ij}^k = \rho_k(Gi, Gj)$ and let $R_k = [r_{ij}^k], k = 1, ..., 4$. Then we have the following matrices R_k which represent the ρ_k.

R_1	$G1$	$G2$	$G3$	$G4$	$G5$	R_2	$G1$	$G2$	$G3$	$G4$	$G5$
$G1$	.50	.50	.76	.64	.55	$G1$	.50	.50	.56	.73	.65
$G2$	.50	.50	.76	.64	.55	$G2$	.50	.50	.56	.73	.65
$G3$	.24	.24	.50	.38	.29	$G3$	.44	.44	.50	.67	.59
$G4$	.36	.36	.62	.50	.41	$G4$	.27	.27	.33	.50	.42
$G5$	.45	.45	.71	.59	.50	$G5$	.35	.35	.41	.58	.50
R_3	$G1$	$G2$	$G3$	$G4$	$G5$	R_4	$G1$	$G2$	$G3$	$G4$	$G5$
$G1$	.50	.50	.69	.76	.66	$G1$	.50	.50	.52	.55	.55
$G2$	.50	.50	.69	.76	.66	$G2$	.50	.50	.52	.55	.55
$G3$	.31	.31	.50	.57	.47	$G3$	.48	.48	.50	.53	.53
$G4$	.24	.24	.43	.50	.40	$G4$	.45	.45	.48	.50	.50
$G5$	.34	.34	.53	.58	.50	$G5$	.45	.45	.48	.50	.50

4. Cores

The core is defined to be he set of undominated options that are not eliminated in a pairwise comparison by a required majority. We can interpret this to mean goals that are not considered weak. We next present the procedure to determine the core.

Let

$$h_{ij}^k = \begin{cases} 1 \text{ if } r_{ij}^k < .5, \\ 0 \text{ otherwise,} \end{cases}$$

where $k = 1, ..., 4$ and $i, j = 1, ..., 5$. Let $H_k = [h_{ij}^k], k = 1, ..., 4$.

H_1	$G1$	$G2$	$G3$	$G4$	$G5$	H_2	$G1$	$G2$	$G3$	$G4$	$G5$
$G1$	0	0	0	0	0	$G1$	0	0	0	0	0
$G2$	0	0	0	0	0	$G2$	0	0	0	0	0
$G3$	1	1	0	1	1	$G3$	1	1	0	0	0
$G4$	1	1	0	0	1	$G4$	1	1	1	0	1
$G5$	1	1	0	0	0	$G5$	1	1	1	0	0

H_4	$G1$	$G2$	$G3$	$G4$	$G5$	H_2	$G1$	$G2$	$G3$	$G4$	$G5$
$G1$	0	0	0	0	0	$G1$	0	0	0	0	0
$G2$	0	0	0	0	0	$G2$	0	0	0	0	0
$G3$	1	1	0	0	1	$G3$	1	1	0	0	0
$G4$	1	1	1	0	1	$G4$	1	1	1	0	0
$G5$	1	1	0	0	0	$G5$	1	1	1	0	0

The extent to which expert k is not against goal Gj is given by the formula

$$h_j^k = \frac{1}{4} \sum_{i=1}^{5} h_{ij}^k, k = 1, ..., 4; j = 1, ..., 5.$$

The extent to which all individuals are not against goal Gj is given by the formula

$$h_j = \frac{1}{4} \sum_{k=1}^{4} h_j^k, j = 1, ..., 5.$$

We assume that a required majority is imprecisely defined, for example is given by a fuzzy linguistic modifier 'most' introduced in [5]. Let Q denote the linguistic modifier 'most.' Let $\mu_Q : [0,1] \to [0,1]$ be defined by all $x \in [0,1]$,

$$\mu_Q(x) = \begin{cases} 1 \text{ if } x \geq .8, \\ 2x - .6 \text{ if } .3 < z < .8, \\ 0 \text{ if } x \leq .3. \end{cases}$$

The extent to which most experts are not against goal Gj is given by the formula,

$$\nu_Q^j = \mu_Q(h_j), j = 1, ..., 5.$$

The fuzzy Q-core is defined to be the fuzzy subset

$$C_Q = \nu_Q^1/G1 + ... + \nu_Q^5/G5,$$

i.e., a fuzzy subset of the set of goals that are not defeated by most experts. We have the following table for the h_j^k :

$j =$	1	2	3	4	5
h_j^1	.75	.75	0	.25	.5
h_j^2	.75	.75	.5	0	.25
h_j^3	.75	.75	.25	0	.5
h_j^4	.75	.75	.5	0	0

We have the following table for the h_j and the ν_Q^j

$j =$	1	2	3	4	5	$j =$	1	2	3	4	5
h_j	.75	.74	.31	.06	.31	ν_Q^j	.9	.9	.02	0	.02

The fuzzy Q-core is thus

$$C_Q = .9/G1 + .9/G2 + .02/G3 + 0/G4 + .02/G5$$

Thus we see that goals 1 and 2 are not dominated (not considered weak) by most experts.

We next consider the strength of defeat. Let

$$\widehat{h_{ij}^k} = \begin{cases} 2(.5 - r_{ij}^k) \text{ if } r_{ij}^k < .5, \\ 0 \text{ otherwise.} \end{cases}$$

The extent to which all individuals are not strongly against goal Gj is given by the formula

$$\widehat{\nu_Q^j} = \mu(\widehat{h_j}), j = 1, ..., 5.$$

The fuzzy s/Q-core is a fuzzy subset of the set of goals that are not strongly dominated by most experts and is defined to be the fuzzy subset

$$C_{s/Q} = \widehat{\nu_Q^1} G1 + ... + \widehat{\nu_Q^5} G5.$$

Let $\widehat{H_k} = [\widehat{h_{ij}^k}], j = 1, ..., 4$.

H_1	$G1$	$G2$	$G3$	$G4$	$G5$	H_2	$G1$	$G2$	$G3$	$G4$	$G5$
$G1$	0	0	0	0	0	$G1$	0	0	0	0	0
$G2$	0	0	0	0	0	$G2$	0	0	0	0	0
$G3$	.52	.52	0	.24	.42	$G3$	.12	.12	0	0	0
$G4$	.28	.28	0	0	.18	$G4$	.46	.46	.34	0	.06
$G5$	.1	.1	0	0	0	$G5$	.30	.30	.18	0	0

H_4	$G1$	$G2$	$G3$	$G4$	$G5$	H_2	$G1$	$G2$	$G3$	$G4$	$G5$
$G1$	0	0	0	0	0	$G1$	0	0	0	0	0
$G2$	0	0	0	0	0	$G2$	0	0	0	0	0
$G3$	.38	.38	0	0	.06	$G3$	.04	.04	0	0	0
$G4$	.52	.52	.14	0	.20	$G4$	.10	.10	.04	0	0
$G5$	.32	.32	0	0	0	$G5$	.10	.10	.04	0	0

We have the following table for the $\widehat{h_j^k}$.

$j =$	1	2	3	4	5
$\widehat{h_j^1}$	.22	.22	0	.06	.15
$\widehat{h_j^2}$	.22	.22	.13	0	.04
$\widehat{h_j^3}$	.30	.30	.03	0	.06
$\widehat{h_j^4}$	.06	.06	.02	0	0

We have the following table for the $\widehat{h_j}$.

$j =$	1	2	3	4	5
$\widehat{h_j}$	.2	.2	.04	.01	.06

We have that the $\widehat{\nu_Q^j}$ has the value 0 for $j = 1, ..., 5$. Thus we cannot conclude that most experts are not against any particular goal.

5. Consensus Winners

We now consider consensus. we follow the scheme $\{R_1, R_2, R_3, R_4\} \to R \to$ solution. that is, from the individual fuzzy preference relations, we determine first a special fuzzy preference relation R which is similar to its individual counter parts, but concerns the whole group of experts. We then find a solution from it. We skip the first part $\{R_1, R_2, R_3, R_4\} \to R$ and assume $R = [r_{ij}]$ is given by

$$r_{ij} = \begin{cases} \frac{1}{4}\sum_{k=1}^{4} a_{ij}^k \text{ if } i \neq j. \\ 0 \text{ otherwise,} \end{cases}$$

where

$$a_{ij}^k = \begin{cases} 1 \text{ if } r_{ij}^k > .5, \\ 0 \text{ otherwise,} \end{cases}$$

$k = 1, 2, 3, 4$.

Consider the second step, $R \to$ solution, i.e., how to determine a solution from a fuzzy preference relation. One solution concept is the consensus winner which will be extended here under a majority expressed by a linguistic modifier.

Let

$$g_{ij} = \begin{cases} 1 \text{ if } r_{ij} > .5, \\ 0 \text{ otherwise,} \end{cases}$$

which expresses whether or not Gi is preferred to Gj. Then

$$g_i = \frac{1}{4} \sum_{j=1}^{5} g_{ij}$$

is the mean degree to which Gi is preferred to $Gj, j = 1, ..., 5$. Also,

$$z_Q^i = \mu_Q(g_i)$$

is the extent to which Gi is preferred to Q other $Gj, j = 1, ..., 5$.

The fuzzy Q-consensus winner is defined to be the fuzzy subset

$$W_Q = z_Q^1/G1 + ... + z_Q^5/G5$$

as the fuzzy subset of the Gi that are preferred to Q other Gj. Let $A_k = [a_{ij}^k], k = 1, ..., 4$. We do not include these matrices here. The A_k matrices yield the matrices $R = [r_{ij}]$ and $G = [g_{ij}]$.

R	$G1$	$G2$	$G3$	$G4$	$G5$
$G1$	0	0	1	1	1
$G2$	0	0	1	1	1
$G3$	0	0	0	.75	.5
$G4$	0	0	.25	0	0
$G5$	0	0	.5	.75	0

G	$G1$	$G2$	$G3$	$G4$	$G5$
$G1$	0	0	1	1	1
$G2$	0	0	1	1	1
$G3$	0	0	0	1	0
$G4$	0	0	0	0	0
$G5$	0	0	0	1	0

We have the following table for the g_i and z_Q^i

$i =$	1	2	3	4	5
g_i	.75	.75	.25	0	.25
z_Q^i	.90	.90	0	0	0

The fuzzy Q-consensus winner is as follows:

$$W_Q = .9/G1 + .9/G2.$$

Hence the goals $G1$ and $G2$ are preferred to most goals.

We now consider the strength of preference. Define

$$\widehat{g_{ij}} = \begin{cases} 2(r_{ij} - .5) \text{ if } r_{ij} > .5. \\ 0 \text{ otherwise.} \end{cases}$$

Then replacing g_i and z_Q^i by $\widehat{g_{ij}}$ and $\widehat{z_Q^i}$ in the above formulas, respectively, we define the fuzzy s/Q-consensus winner as

$$W_{s/Q} = \widehat{z_Q^1}/G1 + ... + \widehat{z_Q^5}/G5.$$

Let $\widehat{G} = [\widehat{g_{ij}}]$. Then

$\widehat{G}$	$G1$	$G2$	$G3$	$G4$	$G5$
$G1$	0	0	1	1	1
$G2$	0	0	1	1	1
$G3$	0	0	0	.5	0
$G4$	0	0	0	0	0
$G5$	0	0	0	.5	0

and

$$\widehat{g}_1 = .75, \widehat{g}_2 = .75, \widehat{g}_3 = .125, \widehat{g}_4 = 0, \widehat{g}_5 = .125.$$

Thus

$$\widehat{g_Q^1} = .9, \widehat{g_Q^2} = .9, \widehat{g_Q^3} = 0, \widehat{g_Q^4} = 0, \widehat{g_Q^5} = 0.$$

Hence the fuzzy s/Q-consensus winner is

$$W_{s/Q} = .9/G1 + .9/G2.$$

6. Degree of Consensus

We first consider the degree of strict agreement between experts m and $n, m, n = 1, ..., 4$ as to their preferences between Gi and $Gj, i, j = 1, ..., 5$.

Define

$$\nu_{ij}(m,n) = \begin{cases} 1 \text{ if } |r_{ij}^m - r_{ij}^n| \leq .05, \\ 0 \text{ otherwise,} \end{cases}$$

$m, n = 1, ..., 4$. Let $\nu(m,n) = [\nu_{ij}(m,n)], m, n = 1, ..., 4$. The matrices are not presented here.

The degree of agreement between individuals m and n as to their preferences of Gi and Gj is

$$\nu_B(m,n) = \frac{1}{10} \sum_{i=1}^{4} \sum_{j=i+1}^{5} \nu_{ij}(m,n),$$

where $m, n = 1, ..., 4$. Thus we have

(m,n)	$(1,2)$	$(1,3)$	$(1,4)$	$(2,3)$	$(2,4)$	$(3,4)$
$\nu_B(m,n)$	.2	.2	.3	.6	.3	.2

The degree of agreement between all individuals is given by the formula

$$\nu^B = \frac{1}{6} \sum_{m=1}^{3} \sum_{n=m+1}^{4} \nu_B(m,n).$$

Thus we have $\nu^B = .3$.

7. How Well Are Universities Doing in Achieving the Overarching Goal

We asked two experts to score how well they felt universities were doing in achieving the goals, $G, G1 - G6$. One expert was a faculty member in the Excercise Science Department and another was from the Residence Life Program. The following matrices provide the experts' opinions on a scale of 1 to 10 with 10 the highest possible score.

Creighton University	G	$G1$	$G2$	$G3$	$G4$	$G5$
$E1$	3	1	1	5	5	3
$E2$	4	2	1	4	6	8

All Universities	G	$G1$	$G2$	$G3$	$G4$	$G5$
$E1$	5	5	5	5	5	5
$E2$	5	3	4	5	5	6

We next substitute the opinions of the experts concerning how well universities are doing in achieving the overarching goal into the appropriate equation. we then compare this result with their original opinions. For example, $E1's$ scores for $G1, G2, G3, G4, G5$ for Creighton are $3, 1, 1.5, 5, 3$, respectively. When these numbers are substituted into the equation, $G = .28G1 + .28G2 + .15G3 + .11G4 + .18G5$, we obtain the value 2.4. This value compares to the value 3 which $E1$ scored for G.

Guiasu and AHP

Creighton	G	$.28G1 + .28G2 + .15G3 + .11G4 + .18G5$
$E1$	3	2.4
$E2$	4	4.98

All Universities	G	$.28G1 + .28G2 + .15G3 + .11G4 + .18G5$
$E1$	5	5
$E2$	5	4.34

Dempster-Shafer

Creighton	G	$.45G1 + .45G2 + .01G3 + .01G4 + .07G5.$
$E1$	3	1.21
$E2$	4	2.01

All Universities	G	$.45G1 + .45G2 + .01G3 + .01G4 + .07G5.$
$E1$	5	5
$E2$	5	3.67

There are a few reasons that universities in general are doing a better job of performing their freshman about the risks and factors that influence weight gain the freshman year of college in comparison to Creighton University. The requirement of a class emphasizing physical activity, diet and balance in life, by freshman at many universities allows the information to reach the ears of the freshman. In addition, the Residence Life programs at other universities inform their students of the effects in different ways. Lastly, the fitness facilities at some colleges and universities are open more hours and offer greater variety than Creighton University's physical fitness facilities provide.

Conclusion

We developed methods to determine how well universities in the US were doing in controlling weight gain of freshmen. We then used expert opinion to run the models. We found that universities scored a 5 out of 10. From this score, we determine that universities have room for improvement.

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Appendix A

To support a program that would decrease the risk for weight gain among college freshman, the following factors should be under primary consideration. The way these factors are denoted is the first number of the subscript indicates the goal it supports and the second indicates which factor number it is in support of the goal.

$G11$: The types of food offered at the dining facilities as well as the number of choices.

$G12$: The cooking ability and facilities available to students outside the dining halls.

$G13$: The cost of food that supports a well balanced and nutritious diet.

$G14$: Fast food and junk food consumption as snacks in between meals provided by the college or university via the dining halls.

$G15$: Amount of food consumed and metabolism that an individual college student experiences and depends on.

$G21$: Access to a fitness facility in close proximity to the residence of the student.

$G22$: Level of motivation to stay fit and healthy within the college student's schedule.

$G23$: Lack of structured athletic program that was present in high school such as an athletic team that provided the motivation by means of a coach and scheduled, mandatory practice regimens.

$G24$: Time available or allotted to focus on physical activity and exercise.

$G25$: Practice of a training group or partner to encourage participation.

$G31$: Class and homework schedules that take up so much of the student's time and energy that they do not have the focus or ability to participate in physical activity.

$G32$: Various clubs, work, volunteer activities and sleep that take up a large amount of a student's non class time throughout a typical day.

$G33$: Lack of a defined schedule to work out that the person insists on keeping to no matter what comes up.

$G34$: Social activities that often replace times that could be used to exercise throughout the busy week of a first year college student.

$G35$: Time constraints associated with having balanced meals and overall balanced diet throughout the day.

$G41$: The feeling that other things are more important than exercise and nutrition so they must be completed first. If exercise and nutrition fall by the wayside it is alright as long as the other activities were completed.

$G42$: High stress levels cause retention of fat in the human body.

$G43$: Use of fattening and sugary foods as comfort foods during periods of high stress or late hours of studying.

$G51$: One's body image as an indicator as to whether or not they need to be exercising frequently or do not think that is necessary.

$G52$: Increase party attendance which often includes high fat and calorie snacks and large amounts of alcohol.

$G53$: Influence of peers on dietary and exercise habits throughout a college student's life.

$G54$: Affect of relationships both on self image, time and priority within a student's life at college during their freshman year.

Appendix B

	$E1$	$E2$	$E3$	$E4$	Row Avg
$G11$	10	3	6	8	$6.75 = w_1$
$G12$	5	4	1	7	$4.25 = w_2$
$G13$	8	8	1	7	$6.00 = w_3$
$G14$	8	10	8	7	$8.25 = w_4$
$G15$	8	10	10	8	$9.00 = w_5$
Column Total	39	35	26	37	34.25

	$E1$	$E2$	$E3$	$E4$	Row Avg	Row Product
$G11$	.26	.09	.23	.22	.20	.0011840
$G12$	.13	.11	.04	.19	.12	.0001086
$G13$	.21	.23	.04	.19	.17	.0003670
$G14$	.21	.29	.31	.19	.25	.0035870
$G15$	.21	.29	.39	.22	.28	.0052252

Guiasu method:

$$G1 = .2G11 + .12G12 + .17G13 + .25G14 + .28G15.$$

Analytic Hierarchy Process:

$$G1 = .2G11 + .12G12 + .18G13 + .24G14 + .26G15.$$

Dempster-Shafer method:

$$G1 = .11G11 + .01G12 + .04G13 + .34G14 + .50G15.$$

	$E1$	$E2$	$E3$	$E4$	Row Avg
$G21$	1	10	1	7	$4.75 = w_1$
$G22$	8	10	10	8	$9.00 = w_2$
$G23$	5	4	5	7	$5.25 = w_3$
$G24$	1	7	3	6	$4.25 = w_4$
$G25$	3	5	3	6	$4.25 = w_5$
Column Total	18	36	22	34	

	$E1$	$E2$	$E3$	$E4$	Row Avg	Column Product
$G21$	.06	.28	.05	.21	.15	.0001764
$G22$	.44	.28	.45	.24	.35	.0133056
$G23$	.28	.11	,23	.21	.21	.0014876
$G24$	.06	.29	.14	.18	.17	.0004384
$G25$	.17	.14	.14	.18	.16	.0005997

Guiasu method:

$$G2 = .15G21 + .35G22 + .21G23 + .17G24 + .16G25.$$

Analytic Hierarchy Process:

$$G2 = .17G21 + .33G22 + .19G33 + .15G34 + .15G25.$$

Dempster-Shafer method:

$$G2 = .01G21 + .83G22 + .09G23 + .03G24 + .04G25.$$

	$E1$	$E2$	$E3$	$E4$	Row Avg
$G31$	1	3	5	7	$4.00 = w_1$
$G32$	1	6	3	7	$4.25 = w_2$
$G33$	1	10	8	6	$6.25 = w_3$
$G34$	5	5	3	7	$5.00 = w_4$
$G35$	5	2	1	7	$3.75 = w_5$
Column Total	13	26	20	34	

	$E1$	$E2$	$E3$	$E4$	Row Avg	Column Product
$G31$	.08	.12	.25	.21	.17	.0005040
$G32$	.08	.23	.15	.21	.17	.0005796
$G33$	.08	.38	.40	.18	.26	.0021888
$G34$	.38	.19	.15	.21	.23	.0022743
$G35$	.38	.08	.05	.21	.18	.0003192

Guiasu method:

$$G3 = .17G31 + .17G32 + .26G33 + .23G34 + .18G35.$$

Analytic Hierarchy Process:

$$G3 = .17G31 + .18G32 + .27G33 + .22G34 + .16G35.$$

Dempster-Shafer method:

$$G1 = .109G11 + .10G12 + .37G13 + .39G14 + .05G15.$$

	$E1$	$E2$	$E3$	$E4$	Row Avg
$G41$	5	10	5	7	$6.75 = w_1$
$G42$	1	3	2	7	$3.25 = w_2$
$G43$	8	8	8	6	$7.50 = w_3$
Column Total	14	21	15	20	17.50

	$E1$	$E2$	$E3$	$E4$	Row Avg	Column Product
$G41$	.36	.48	.33	.35	.38	.0199584
$G42$	.07	.14	.13	.35	.17	.0004459
$G43$	.57	.38	.53	.30	.45	.0344394

Guiasu method:

$$G4 = .38G41 + .17G42 + .45G43.$$

Analytic Hierarchy Process:

$$G4 = .39G41 + .19G42 + .43G43.$$

Dempster-Shafer method:

$$G4 = .37G11 + .01G42 + .63G43.$$

	$E1$	$E2$	$E3$	$E4$	Row Avg
$G51$	5	8	7	7	$6.75 = w_1$
$G52$	9	10	10	7	$9.00 = w_2$
$G53$	8	10	8	6	$8.00 = w_3$
$G54$	5	7	6	6	$6.00 = w_4$
Column Total	27	35	31	26	29.75

	$E1$	$E2$	$E3$	$E4$	Row Avg	Row Product
$G51$	.19	.23	.23	.27	.23	.0027137
$G52$	.33	.29	.32	.27	.30	.0082684
$G53$	.30	.29	.26	.23	.27	.0052026
$G54$	.19	.20	.19	.23	.20	.0016606

Guiasu method:

$$G5 = .23G51 + .3G52 + .27G53 + .2G54.$$

Analytic Hierarchy Process:

$$G5 = .23G51 + .3G52 + .27G53 + .2G54.$$

Dempster-Shafer method:

$$G5 = .15G51 + .46G52 + .29G53 + .09G54.$$

Making Sense of Nuclear Deterrence in the Post-Cold War Era: A Formal Approach

Carly Goodman, Karly Hermanson, and Anne Marascuilo
Creighton University

The dramatic shift in the global security environment that followed the collapse of the Soviet Union and the end of the Cold War has been attended by an explosion of terms in the vocabulary of nuclear deterrence. During the Cold War, nuclear deterrence focused almost exclusively on deterring the Soviet Union from using nuclear weapons. In contrast, the current international environment demands equal attention be given to reducing the threat of the spread of nuclear materials, weapons, and weapons technology (Bunn 2007; DoD 2006; McKenna 2008; Payne 2000; Pilat 2009). Unfortunately, while many of the newly coined terms capture important nuances, they appear unconnected to each other, thereby discouraging analysts and scholars from seriously considering any insights they might contain.

We argue in this paper that the vocabulary of nuclear deterrence, henceforth simply referred to as deterrrence, need not be as difficult to understand nor its terms as disconnected as they appear. We proceed in our task with the contention that deterrence is a strategy intended to achieve the goal of nuclear stability and that the number of dimensions on which global nuclear stability is threatened in the post-Cold War era has increased. Taking our lead from the literature, we understand deterrence as an effort to effect the cost-benefit utility calculation of an adversary in order to convince him from taking an unwanted action. In other words, deterrence inolves the manipulation of utility calculations. We then present a directed acyclic graph (DAG) identifying the decisions that state and non-state actors face that present a potential threat to global nuclear stability. These decisions represent nodes at which the U.S. must intervene should the utility calculation of the state or non-state actor otherwsie favor behavior undermining nuclear stability. Next we turn to the heart of our argument. Each node represents a different deterrence strategy. We formally define each of these strategies and conclude by considering the logical relationships between them.

Our approach is formal. We contend that in the absence of rigorously formal definitions reflecting clearly identified assumptions, the deterrence community will remain unable to engage in the kind of analytical thinking necessary to formulate a theory of deterrence that can inform U.S. policy in the current international context. The clarity with which a formal approach does so will invite easy criticism. We do not shy away from such criticism, but we invite our critics to be as clear about the set of assumptions that they prefer as we are. If they are, then we (and they) can re-define and uncover relationships to see if the new assumptions result in any substantive difference.

1 Deterrence and Rational Actors

We understand deterrence to be a strategy intended to achieve the goal of nuclear stability. During the Cold War, nuclear stability was understood as a status quo situation in which neither the U.S. nor the Soviet Union would employ nuclear weapons. Hence, deterrence was singularly focused on reducing the likelihood of a Soviet first-strike (Garthoff 1990; Schelling 1960; Smoke & George 1973; Waltz 1990). Often modeled in the form of games with rational actors, deterrence was understood as an attempt to convince an adversary not to engage in an action (in the context of the Cold War, a nuclear first-strike) by threatening to impose unacceptably high costs for doing so (Jervis 1989; Schelling 1960). In order to be successful, the threat had to be credible (Schelling 1960; Smoke & George 1973; Waltz 1990).

The post-Cold War global environment presents a broader menu of threats to nuclear stability (Bunn 2004; CSIS 2007; DoD 2006a, 2006b, 2010; Garthoff 1990; Hayes & Wheatley 1996; Jervis 1989; Kahan 1996; Lamb et al. 2005; Nye 2009; Perry 1996; Schwartz & Kirk 2009). Not only must states resist employing nuclear weapons, but stability requires that the number of states with nuclear weapons not increase. Moreover, non-state actors must also not come into possession of a nuclear capability. The non-state actors with which the international community is most concerned in relation to nuclear stability are terrorists groups (Allison 2008; Caplin 2006; DoD 2010; Frederking 2003; Lantis 2009; Perkovich 2009; Schaub 2009).

We understand nuclear stability in the post-Cold War system as a condition in which states not only refrain from the use of nuclear weapons, but they neither acquire nor proliferate nuclear materials, weapons, or technology. We define use to include the full arming, launching, or releasing of a nuclear weapon. This includes the testing of nuclear weapons.[1] The term acquiring encompasses obtaining or gaining possession of any materials or technology in relation to nuclear delivery systems, fuel cells, and facilities. Proliferating is defined as the intentional contribution, access to or sale of nuclear weapons to an external state actor. Moreover, the broader definition of nuclear stability in the post-Cold War era further requires that states cooperate in reducing the proliferation and acquisition of nuclear materials, technology, and weapons. It is not enough that they themselves refrain from engaging in such activity, but they must also pro-actively assist in efforts to deny others from doing so. Thus, we understand cooperation to mean a willingness by a state to allow inspections from the international community of all nuclear facilities to determine their non-nuclear weapon purpose.[2]

These four dimensions of nuclear stability - *use, acquisition, proliferation, and cooperation* - each require a conscious decision on the part of state and

[1] See DOD Dictionary of Military Terms, available: http://www.dtic.mil/doctrine/dod_dictionary

[2] Our definitions for acquire, proliferate, and cooperate roughly accord with those found in "Nuclear Stability Assessment Framework," Global Innovation and Strategy Center (GISC), United States Strategic Command, 2009.

non-state actors. The post-Cold War literature continues to assume that these actors are rational (Berejikian 2002; Caplan 2006; Chilton & Weaver 2009; Danilovic 2001b; DoD 2006a; Frederking 2003; Huth et al. 1993; James 2000; Jervis 1989; Kraig 1999; McKenna 2008; Payne 2000; Perry 1996; Quackenbush 2006; Schelling 1960; Schwartz & Kirk 2009; Singh & Way 2004; Smoke & George 1973; Zagare & Kilgour 1993, 2009). The assumption carries with it two consequences. First, rational state and non-state actors take actions intended to maximize their utility. Second, in order to be effective, a deterrence strategy must change the cost-benefit utility calculation of a state or non-state actor to favor eschewing behavior that undermines nuclear stability, that is behavior on the four dimensions defining nuclear stability (Danilovic 2001a, 2001b; Huth, et. al. 1993; Signorino & Tarar 2006; Zagare and Kilgour 1993, 2009). While some scholars have questioned the rationality of terrorists, the general view is that they perform a decision calculus based on their leaders' objectives (DoD 2010; Lantis 2009; Schaub 2009). While the consequence of terrorist leaders acting on the basis of narrow self-interests is that in many cases the costs of irrational actors are born by others (Caplin 2006), the decision-making calculus remains essentially rational. Given that both state and non-state actors can be treated as rational, we henceforth use the term state to denote both.

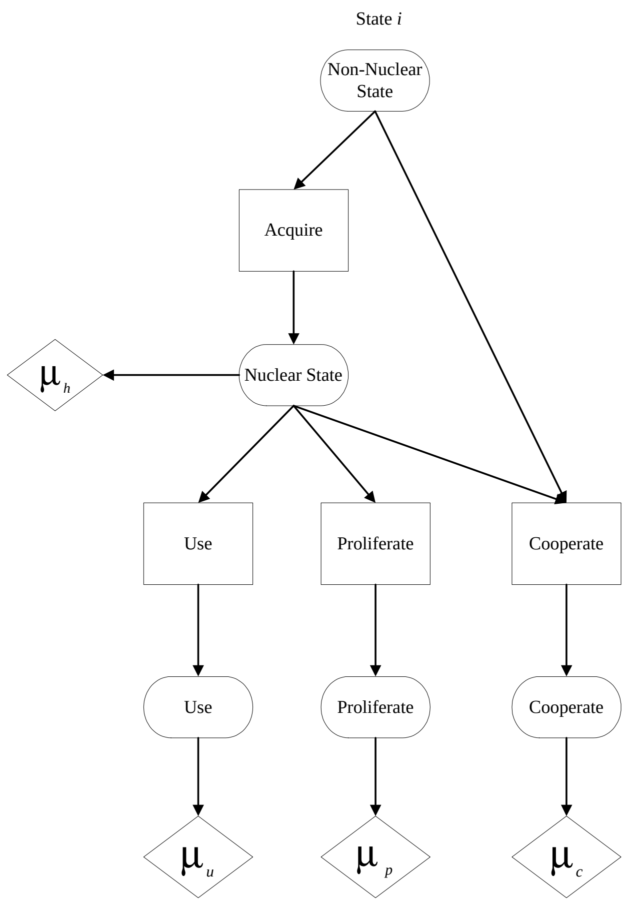
2 Nuclear Stability: A Directed Acyclic Graph

There is a logic that permits us to order the four dimensions of nuclear stability in relation to one another. That logic is presented in the directed acyclic graph (DAG) at Figure 1. DAGs map variables (circles), decision points (rectangles), and utilities (diamonds).

The DAG at Figure 1 identifies the choices faced by the two classes of states in the international system: *nuclear states* and *non-nuclear states.* For the sake of generalization, state i is either in possession of nuclear weapons or it is not. If it is, we label it a nuclear state. A nuclear state is faced with three decisions, whether to use nuclear weapons, whether to proliferate nuclear weapons, or whether to cooperate in denying their spread in the international system. Given that state i is rational, its decision will depend on a comparison of the expected utility from each course of action. A non-nuclear state is faced with two decisions, whether to acquire nuclear weapons or whether to cooperate. Again, its decision is based on a comparison of the expected utility from these two choices. If the expected utility of acquiring nuclear weapons is higher than that from cooperation, it will acquire these weapons and subsequently confront the three choices faced by nuclear states: use, proliferate, or cooperate.

Closer inspection of the DAG at Figure 1 further reveals that the choice faced by a non-nuclear state i between acquistion and cooperation is affected by its calculation of the expected utility from use and proliferation. Thus, a state will not only acquire a nuclear weapon because its expected utility from possessing such weapons is higher than that from cooperation, but it will do so if its utility from using them or proliferating them is higher than that from cooperation

Figure 1



State *i*
Non-Nuclear State
Acquire
Nuclear State
μ_h
Use
Proliferate
Cooperate
Use
Proliferate
Cooperate
μ_u
μ_p
μ_c

(even if its utility from cooperation exceeds that from simple possession).

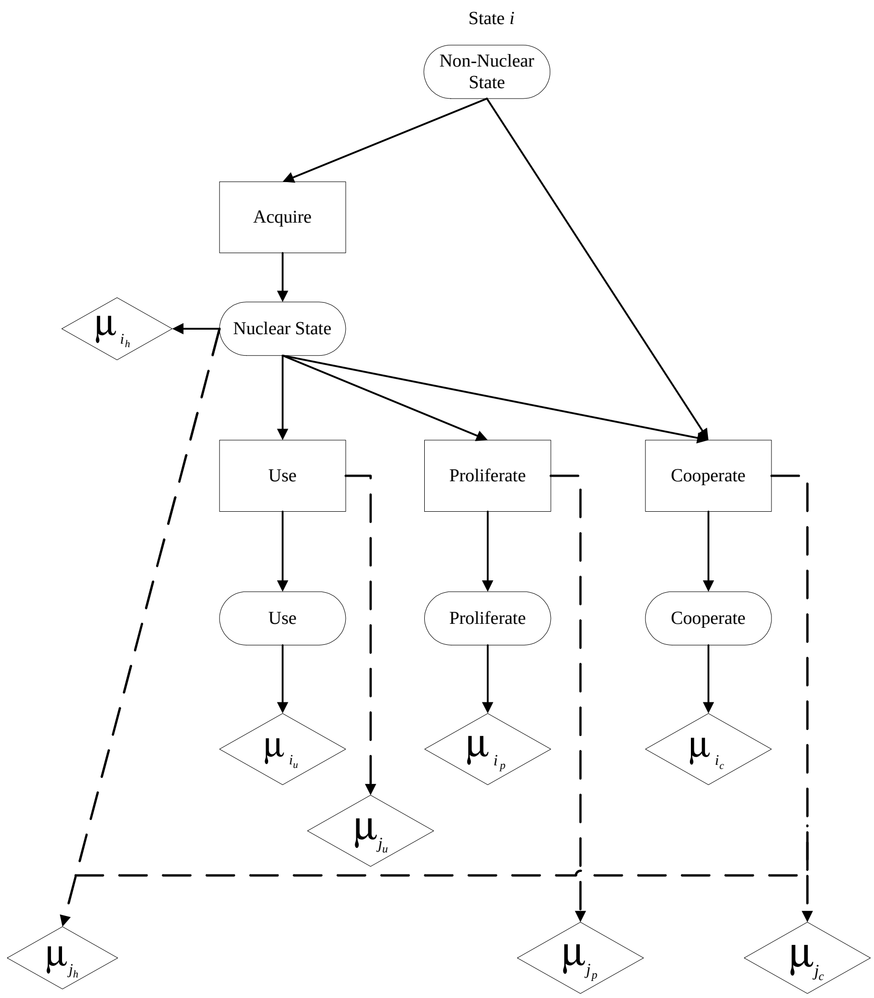
The discerning reader will likely raise the issue of the degree to which choices facing a state are interdependent. That is, does the choice of one option necessarily negate the other from being chosen? Clearly the choice on the part of a nuclear state to cooperate negates the choice to use or proliferate nuclear weapons. By the same logic, the choice of a non-nuclear state to acquire negates the option to cooperate in the first instance. It may only revisit the choice to cooperate after it achieves the status of a nuclear state. However, it is not outside the realim of possibility that a nuclear state might choose to use *and* proliferate nuclear weapons. While this is an intriguing possibility, it has little affect on the conclusions of our argument. As we will demonstrate, the point of deterrence is to reduce the likelihood of either. Moreover, since both use and proliferation undermine nuclear stability, we might just as well treat them as a single choice: *not to cooperate.* We choose not to do so in order to clarify the difference in terms in the deterrence vocabulary.

Of course, state i depicted in Figure 1 is not operating in a vacuum. Indeed, the utility calculations for each decision are impacted by some combination of the situations of the other states in the international system For ease of representation and without loss of generality, we will refer to the rest of the states in the international system as state j. The directed acyclic graph depicting the interaction of state i and the international sytem, state j, is in Figure 2. The DAG shows that the choices of states affect the utility calculations of others. For instance, if a state uses a nuclear weapon, then it will likely affect the utility of the use of a nuclear weapon by other states. Furthermore, if a state acquires a weapon, it will affect the utility calculation of other states relative to acquisition, proliferation, and cooperation. The same is true for proliferation or cooperation. As states engage in either, they affect the utility calculations of other states relative to acquisition, proliferation, and cooperation.

In the absence of U.S. intervention, nuclear stability relies on the mutual interaction of decisions made by states relative to cooperation, aquisition, proliferation, and use as depicted in Figure 2. Were the system of interactions to result in an acceptable degree of nuclear stability, the U.S. would have no reason to intervene. The U.S. goals are that no state use, acquire, or proliferate nuclear weapons and that all states cooperate on reducing their spread.

In order for the U.S. to reduce the likelihood that states will use, acquire, or proliferate nuclear weapons, or to increase the likelihood that a state will cooperate, it must affect the expected utility resulting from these actions. The DAG at Figure 3 shows that the U.S. can affect an actor's utility at any of these four distinct points. Since utility calculations derive from assessments of the costs and benefits of a given action, the U.S. strategy must be to employ some mix of the two to change the player's expected utility to favor outcomes supporting global nuclear stability, that is either to deny use, acquisition or proliferation or to incentivize cooperation. Its strategies for doing so have broadly been referred to in the literature as deterrrence. However, that same literature has suggested numerous terms for what appear to be an array of strategies only loosely, if at all, related to deterrence. These include damage

Figure 2


State i
Non-Nuclear State
Acquire
Nuclear State
μ_{i_h}
Use
Proliferate
Cooperate
Use
Proliferate
Cooperate
μ_{i_u}
μ_{i_p}
μ_{i_c}
μ_{j_u}
μ_{j_h}
μ_{j_p}
μ_{j_c}

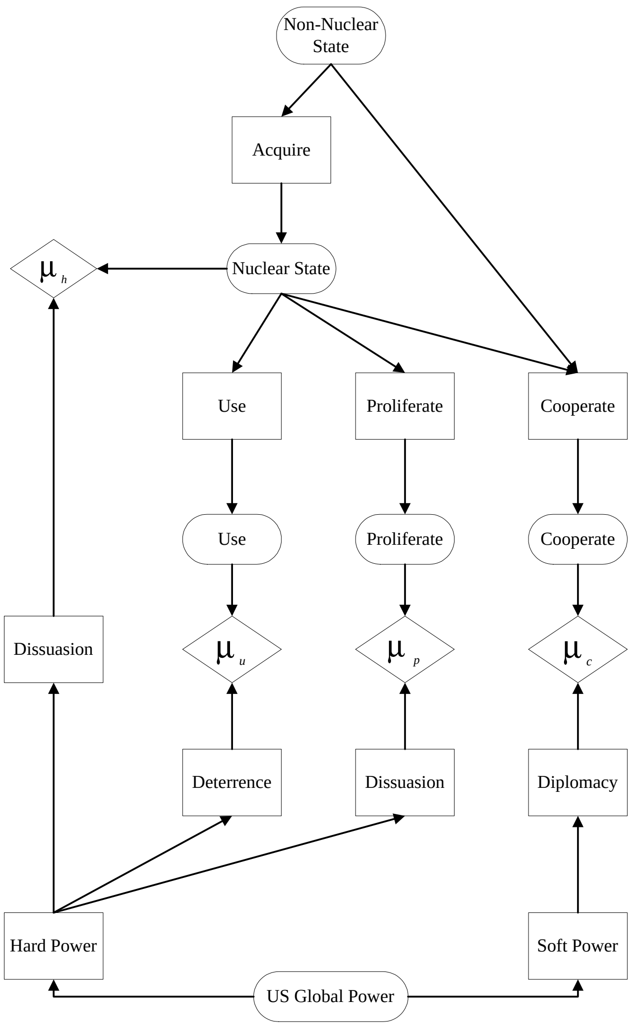
limitation, assurance, insurance, dissuasion, and inducements (see Schwartz & Kirk 2009 for an excellent review of these concepts). In what follows, we demonstrate that each of the points for U.S. intervention noted in Figure 3 are related to a particular deterrence strategy. We formalize these strategies and relate them to one another in order to bring greater conceptual clarity to the field.

Before doing so, we should make note of the fact that some of the conceptual muddle that attaches itself to the deterrence literature owes to the concept of tailored deterrence. Tailored deterrence recognizes that a unilateral deterrence strategy will not work, and expresses the need for individualized strategies (Bunn 2007; Payne 2009a). A multilateral deterrence policy must also consider its second- and third- order effects on other potential adversaries (Chilton & Weaver 2009; DoD 2006a; McKenna 2008). This involves crafting context specific deterrence policies for different adversaries based on their beliefs, national or cultural identity, dominant leaders, and military organization (Lantis 2009; Payne 2000). Thus, the U.S. must put together a package of costs and benefits fitting the uniqueness of each situation (Bunn 2007; Chilton & Weaver 2009; DoD 2006a; Lantis 2009; McKenna 2008; Payne 2000, 2009a; Yost 2009). We do not argue that this is not the case. Our contention is that the focus on tailored deterrence has had the unfortunate, and perhaps unintended, consequence of suggesting that deterrence is an ad hoc concept that defies generalization and can only be dealt with contextually. In our view, costs and benefits are the primitives that provide a unifying basis for all deterrence strategies. It matters not what the particular package of promises or threats on offer is; it is intended to increase the cost or benefit from an action.

Moreover, our general approach explicitly incorporates a number of concepts. One of these is the notion of the benefits of restraint (McKenna 2008). The 2006 Deterrence Operations Joint Operating Concept (DOJOC) labels the components of utility calculation as the benefits of a course of action, the costs of a course of action and the consequences of restraint. The consequences of restraint is a single term comprised of both the benefits and costs of not taking a course of action. We treat the benefit from restraint and the cost of restraint as part of the overall assessment of each respective element. Where the U.S. intervenes to change these calculations we treat them as components of the change to the cost and benefit introduced by the U.S. strategy. In a similar vein, our approach also easily accommodates the notion of a general utility calculation based on an assessment of costs and benefits, which includes a consideration of both the costs imposed after an action is undertaken as well as an adversary's potential gain (Bunn 2004; DoD 2006b, 2010; Yost 2009), and the costs and benefits of non-action (Bunn 2007; Chilton & Weaver 2009; Schaub 2009). We emphasize that the point of our paper is not to commend any particular package of costs and benefits over another. Our aim is to bring better clarity and coherence to the strategy of deterrence.

Figure 3

State *i*



3 The Primitive Components of Deterrence

In Schelling's formulation of deterrence, state k attempts to reduce the likelihood that state i will engage in a nuclear first strike by increasing the costs of its doing so. The need to deter state i is rooted in state $i's$ calculus that there is gain to be had by engaging in a first strike. That gain can be represented as B_i^p, where $B_i^p \in [0, 1]$. Absent any response from state k, state i calculates that it will experience a net gain of B_i^p from a first strike. In order to deter state i from engaging in a first strike, state k engages in a strategy of imposing a cost, C_i^p, where $C_i^p \in [0, 1]$. C_i^p represents the costs imposed by state k on state i that are intended to punish state i for a first strike. Thus, state k affects state $i's$ utility from the use of nuclear weapons. In the context of the Cold War era, the cost imposed, C_i^p, had to exceed or equal the expected gain, B_i^p. Hence, a deterrence strategy is judged sufficient if and only if $C_i^p \geq B_i^p$ (i.e., state i is deterred and a nuclear exchange is averted).

In the post Cold War global environment, the United States is no longer assumed to be faced with a single, fixed opponent bent on using nuclear weapons to defeat the U.S. and thereby forcibly re-arrange the international system. Absent this assumption, states must have a reason to engage in system disrupting activities within the context of the multi-dimensional threat environment of the post-Cold War era. Such activities are no longer limited to the use or threatened use of nuclear weapons, but include their acquisition and proliferation as well. Thus, the new deterrence thinking must address a wider set of threats. Furthermore, it must focus on *both* costs and benefits in order to assess where a threat may emerge and how to deal with it.

For the sake of conceptual consistency, we assume that a state's cost-benefit calculus includes an initial assessment of the gains and losses imposed upon it by the existing international status quo. (Since it was simply assumed that the Soviet Union was anti-systemic during the Cold War, this part of the calculus was rendered moot.) In essence, state i views the current global situation as providing it some benefit B_i, where $B_i \in [0, 1]$, and some cost C_i, where $C_i \in [0, 1]$. (Henceforth, we will refer to actor i as state i, whether it is a state or non-state actor. Our definitions will apply equally to both.) State $i's$ net gain, G_i, from the status quo can be expressed as $G_i = B_i - C_i$.

State i wishes to maximize G_i. When $G_i = 1$, the international system perfectly accords with state i's preferences. Conversely, when $G_i = -1$, the international system is perfectly discordant with state i's preferences. It is only when $G_i < 0$ that state i may represent a plausible threat, since state i is dissatisfied with the status quo. One major means for redressing its dissatisfaction is to acquire, proliferate, or threaten the use of, nuclear weapons. While the acquisition or proliferation of nuclear weapons is not the same thing as a first strike, the logic of the cost-benefit analysis is identical: the closer G_i approaches -1, the more likely it will attempt to obtain or use nuclear weapons. In so doing, state i hopes to gain B_i^p. Thus, the net gain is $G_i = B_i - (C_i - B_i^p)$. In words, state i hopes to increase its gain by subtracting the benefits for proliferation from its system costs. The set of states that the U.S. is likely to have to deter are those for whom $G_i > 0$ *only* when B_i^p is added to the cost-benefit equation. That is, the U.S. must deter those states for whom the following two equations hold:

$$\begin{array}{l} G_i = [B_i - C_i < 0] \\ G_i = [B_i - (C_i - B_i^p) > 0] \end{array} ,$$

where

$$G_i = \begin{cases} 1 \text{ if } Gi \geq 1 \\ -1 \text{ if } Gi \leq -1 \\ G_i \text{ otherwise} \end{cases} .$$

In the new, multi-dimensional threat environment of the post-Cold War era, these actors include the so-called rogue states as well as transnational terrorist groups such as al Qaeda and Hezbollah, which seek to acquire nuclear weapons. What makes this group of actors particularly troubling is that there appears to be a reasonable expectation that should they acquire such weapons they are likely to use them.

The cost-benefit equation $G_i = B_i - (C_i - B_i^p)$ carries with it several implications for deterrence. First, in order to adjust the outcome of the cost-benefit analysis so that $G_i > 0$ - by some other means than state i acquiring, proliferating, or using nuclear weapons - the United States must offer additional costs and/or benefits. One way to do this is to change the initial cost-benefit calculus solely on the basis of system factors (B_i & C_i) by working to change the global benefits (public goods) provided to states by the international order. This appears to be the rationale of the thinking behind *Smart Power* (Center for Strategic and International Studies 2007). Another way to accomplish this is to directly threaten or offer private rewards to a particular state. While both approaches focus on the cost-benefit analysis of states, the latter is of the most immediate concern to us in this paper. It is the approach on which deterrence theory focuses.

The second implication of the cost-benefit equation $G_i = B_i - (C_i - B_i^p)$ is that the exact mix of threats or private benefits offered can be specific to the given situation or actor to be deterred. This is what has been labeled as *tailored deterrence* (Bunn 2007, Yost 2008). In essence, the U.S. is attempting to further manipulate state $i's$ calculation of G_i. Thus, if $G_i = [B_i - (C_i - B_i^p) > 0]$, where $G_i = [B_i - C_i < 0]$, then the U.S. may add costs or benefits to attempt to force a reassessment of the net gain to state i. If the U.S. adds costs for proliferation, they are expressed as $C_i^p \in [0, 1]$. These private costs are subtracted from the benefit from proliferation. If the U.S. adds private benefits for not engaging in proliferation, they are expressed as $B_i^{\tilde{}p} \in [0, 1]$. These benefits are added to the benefits from the international system. Formally, the U.S. has rearranged the formula so that

$$G_i = [[(B_i + B_i^{\tilde{}p}) - (C_i - (B_i^p - C_i^p))] \geq 0] ,$$

which can be further simplified as

$$G_i = [[(B_i + \tilde{B_i}^p + B_i^p) - (C_i + C_i^p)] \geq 0] \quad ,$$

where

$$(B_i + \tilde{B_i}^p + B_i^p) = \begin{cases} 1 \text{ if } (B_i + \tilde{B_i}^p + B_i^p) \geq 1 \\ -1 \text{ if } (B_i + \tilde{B_i}^p + B_i^p) \leq -1 \\ (B_i + \tilde{B_i}^p + B_i^p) \text{ otherwise} \end{cases}$$

and

$$(C_i + C_i^p) = \begin{cases} 1 \text{ if } (C_i + C_i^p) \geq 1 \\ -1 \text{ if } (C_i + C_i^p) \leq -1 \\ (C_i + C_i^p) \text{ otherwise} \end{cases}$$

This is the basic definition of the deterrence challenge in the post-Cold War era. Since the package of benefits and costs offered by the U.S. is tailored, either $\tilde{B_i}^p$ or C_i^p may be equal to 0, though neither need be. (Of course, if the U.S. is attempting to persuade a state, then at least one of these two factors can not be equal to 0.). In the ensuing section, we re-define the vocabulary of deterrence in relation to this basic definition of the deterrence challenge.

4 Implications and Relationships

Deterrence thinking during the Cold War focused solely on manipulating the cost side of the cost-benefit equation $G_i = [(B_i + \tilde{B_i}^p + B_i^p) - (C_i + C_i^p)]$. Thus, the equation was reduced to $G_i = [(B_i + B_i^p) - (C_i + C_i^p)]$. Furthermore, since the Soviet Union was assumed to be at best agnostic concerning the existing international order, the equation was further simplified to $G_i = (B_i^p - C_i^p)$. Thus, the Soviet Union experienced a net gain only when the benefit from first use exceeded the cost imposed by the U.S. for first use, in which case the Soviet Union had every incentive to engage in a first strike. To avoid such a scenario, the U.S. worked to increase the costs of first use, and in so doing keep the net gain to the Soviets below an acceptable level. However, since the Soviet Union was assumed to be inherently aggressive, this was not considered problematic.

The end of the Cold War also ended the assumption that any given state was inherently aggressive. Thus, the inclusion of system costs and benefits (B_i & C_i) was not only possible, but it introduced a more nuanced and fuller conceptualization of the deterrence challenge.

Adding system factors, the cost-benefit calculus of the Soviet Union would have been $G_i = [(B_i + B_i^p) - (C_i + C_i^p)]$. This is essentially the case in which the private benefit offered by the United States to a state for non-use ($\tilde{B_i}^p$) is set to zero in the post-Cold War deterrence equation $G_i = [[(B_i + \tilde{B_i}^p + B_i^p) - (C_i + C_i^p)] \geq 0]$. It represents the approach to deterring Russia and China in the post-Cold War era. In essence, it presents a more refined version of deterrence logic. However, while system factors have been added, the focus of deterrence is still decidedly on adding costs to weigh against the assessed benefits of first use. This is the essence of the deterrence strategy shown in Figure 3 in which the U.S. affects μ_u. Assuming that Russia and China receive a net benefit from the current international system (an assumption that seems relatively safe to make), the addition of costs provides a safety net. (We will forego the temptation to discuss whether this might not have been the case in the Cold War as well.)

In our view, $G_i = [(B_i + B_i^p) - (C_i + C_i^p)]$ represents little more than a nuanced version of the classic deterrence model as elaborated by Schelling and others. A strategy of attempting to persuade a state not to engage in the use of nuclear weapons based on threats is properly labeled *deterrence*. In so labeling the equation $G_i = [(B_i + B_i^p) - (C_i + C_i^p)]$ as such, we are keeping with the use of the term as it originally evolved.

To this point, we have dealt with two of four possible variants of the model $G_i = [(B_i + B_i^{\tilde{p}} + B_i^p) - (C_i + C_i^p)]$: the case in which both private benefits and private costs are set to zero, in which case no effort is being made by the U.S. to persuade a state one way or the other, and the case in which only private benefits are set to zero, which we have defined as deterrence. (The first case is the preferred option when $G_i = [B_i - C_i > 0]$.) That leaves two other possibilities: the case in which only private costs are set to zero and that in which neither private costs nor benefits is set to zero.

The former situation defines *assurance.* The U.S. seeks to prevent any unwanted action which threatens not only its own interests but those of its allies (Bunn 2004; CSIS 2007; DoD 2006a, 2006b, 2010; Garthoff 1990; Hayes & Wheatley 1996; Kahan 1996; Lamb et al. 2005; Nye 2009; Perry 1996; Schwartz & Kirk 2009). By assuring allies against attacks by a third party, the U.S. hopes to convince them not to obtain their own nuclear weapons (Bunn 2007; Berejikian 2002; Danilovic 2001a; Pilat 2009). This applies to countries such as Japan and South Korea that the United States could hardly hope to threaten, but must instead convince that it is able to provide sufficient protection (assurance) against a nuclear strike. Assurance is defined as $G_i = [(B_i + B_i^{\tilde{p}} + B_i^p) - (C_i)]$, in which G_i can be less than 0 when C_i increases. This can occur when systemic threats to state i emerge. The equation calls for the U.S. to increase $B_i^{\tilde{p}}$ in order to avert state i from engaging in proliferation in a self-help effort to raise B_i^p. Essentially, the U.S. seeks to raise private benefits $B_i^{\tilde{p}}$ to a level sufficient to disincentivize state i from seeking to raise B_i^p. Assurance largely involves the use of soft power and attempts to affect the utillity calculation of a player's possessing nuclear weapons (μ_h in Figure 3).

The fourth variant involves the complete equation $G_i = [(B_i + B_i^{\tilde{p}} + B_i^p) - (C_i + C_i^p)]$. This defines *dissuasion*. A successful dissuasion strategy requires that the benefit from the international status quo to state i together with the package of benefits and costs proffered by state j exceed the sum of the cost from the international status quo and the perceived benefit of proliferating nuclear weapons to state i. The U.S. may pursue this strategy when trying to stop an actor from acquiring, or proliferating nuclear weapons. Thus, the U.S. is affecting the utility calculation μ_h and μ_p in Figure 3.

5 Dissuasion

We now demonstrate that dissuasion is the overarching concept for the post-Cold War vocabulary regarding non-proliferation and denial of first-use. We begin by noting that the assessment of the costs (C_i) and benefits (B_i) from the system are fixed. Hence, they are not relevant to distinguishing between dissuasion (Ds), deterrence (Dt), and assurance (A). The essential core of assurance is its relationship to private benefits from non-proliferation. Thus,

$$A \iff (\tilde{B_i}^p) \quad .$$

Deterrence focuses on private threats. Thus,

$$Dt \iff C_i^p \quad .$$

Formally, dissuasion is

$$Ds \iff (\tilde{B_i}^p) \text{ or } (C_i^p) \text{ or } (\tilde{B_i}^p \;\&\; C_i^p) \quad .$$

From which it follows that,

$$\begin{array}{l} Ds \iff (A) \text{ or } (Dt) \text{ or } (A \;\&\; Dt) \\ Dt \text{ implies } Ds \\ A \text{ implies } Ds \end{array} \quad .$$

Hence, deterrence and assurance are special cases of dissuasion. Dissuasion is the broader category that encompasses both.

Dissuasion encompasses other concepts in post-Cold War deterrence vocabulary as well. The solution offered by some to the problem of how to deter terrorists is to either prevent or defeat their objectives (Perkovich 2009). Prevention may involve defending against the act itself. A deterring actor utilizes a defense strategy (Dfd) to prevent potential threats (Perry 1996). Defense is a strategy intended to deny any benefit from an act of nuclear proliferation (Schelling 1956). It also involves the reduction of the deterrer's costs and risks in the event that deterrence fails (Adams 2003/2004).

Formally,

$$Dfd \iff \tilde{\ } B_i^p \quad .$$

Thus, $G_i = [(B_i + \tilde{B_i}^p) - (C_i + C_i^p)]$.

Another strategic concept is *defeat* (Dft), which occurs when deterrence fails (Garthoff 1990). It implies a denial of all benefits and the imposition of a total cost on state i (Schelling 1958).

Formally,

$$Dft \iff [\tilde{\ } Bpi \;\&\; (C_i^p = 1)] \quad ,$$

from which it follows that

$$Dft \Longrightarrow Dfd \ \& \ Dt \Longrightarrow Dfd \ \& \ Ds \quad ,$$

and $G_i = [(B_i + \tilde{B_i^{p}}) - (C_i + 1)]$. In words, similar to a strategy of defense, a strategy of defeat denies the benefit of proliferation, but it also sets the left hand side of the equation to its maximum. Hence, state (or non-state) actor i has no options for achieving a positive gain from the international system.

6 Further Considerations

To this point, we have discussed the relationship between the key concepts that comprise the vocabulary for deterrence in the post-Cold War era. We have thus far concluded that the new vocabulary of tailored deterrence should be relabeled dissuasion, which as we have logically demonstrated is the overarching concept in the vocabulary. However, we have yet to consider the application of such strategies. Consider, then, an example in which state k wishes to deter the action of proliferation by state i. What is the proper response of state k to the threat of proliferation or first use posed by state i? Specifically, should state k engage in a strategy of dissuasion, deterrence, or assurance?

The logic of the directed acyclic graph (DAG) at Figure 3 argues that the answer to that question depends on the relative magnitude of state $i's$ utility from acquiring (μ_h), using (μ_u), proliferating (μ_p), and cooperating (μ_c). We arbitrarily score utility along a continuum from 0 to 1 and select .5 as the cutoff between a high (1) and low (0) utility on each of these four dimensions. Then there are eight possible permutations representing the choices faced by nuclear states. They are listed in Table 1.

Row	Policy	utility from use (μ_u)	utility from proliferation (μ_p)	utility from cooperation (μ_c)
1	none	$< .5$	$< .5$	$\geq .5$
2	Diplomacy	< 5	$< .5$	$< .5$
3	Dissuasion	$< .5$	$\geq .5$	$\geq .5$
4	Dissuasion and Diplomacy	$< .5$	$\geq .5$	$< .5$
5	Deterrence	$\geq .5$	$< .5$	$\geq .5$
6	Deterrence and Diplomacy	$\geq .5$	$< .5$	$< .5$
7	Deterrence	$\geq .5$	$\geq .5$	$\geq .5$
8	Deterrence	$\geq .5$	$\geq .5$	$< .5$

Table 1
Possible Permutations of the Utility Calculations of Nuclear States

In the first case (row 1), the utility from use and proliferation are low while that from cooperation is high, state k need not follow any course of action to affect state i. State $i's$ utility calculations will lead it to cooperate. However, if as in case 2 the utility on all three dimensions is low, then state k need only focus on raising the utility from cooperation. None of the strategies discussed in this paper would be helpful owing to the fact that they are fundamentally concerned with the application of hard power to dissuade and deter by threats of costs and promises of benefits. While assurance would appear to commend itself, it too is focused on hard military and economic power: the promise that the U.S. will extend its nuclear and conventional protection. Thus, the use of soft power by diplomacy based on moral persuasion is the best route in such instances.

The next two cases represent situations in which state k is faced with the threat of proliferation on the part of state i. In case 3, state k need only engage in dissuasion to lower state $i's$ utility from proliferation. In case 4, state k must add diplomatic meaures to increase state $i's$ utility from cooperation.

The final four cases represent situations in which state i is likely to use nuclear weapons. In the first of these (row 5), state k need only engage in deterrence to lessen the likely of use to levels at which state $i's$ utility from cooperation will lead in this direction. In the second instance (row 6) state k will have to engage in deterrence to lessen the likelihood of use and diplomacy to increase the utility from cooperation. In the third instance (row 7), state k will have to engage in dissuasion to lessen the likelihood of both use and proliferation; and in the final instance (row 8) it must mix dissuasion with diplomacy.

Non-nuclear states present a bit more complex challenge. While the choice they face is to acquire or cooperate, it is possible that state i may choose to acquire weapons even if its uitility from possessing them is low but its utility from either use or proliferation is high. The reader may recall that we raised this issue of the interdependence of choices earlier in the paper. We are now in a position to see that if either the utility from acquiring, using, or proliferating is high, state i will choose to acquire nuclear weapons. Thus, the challenge facing state k is represented by the eight possible permuations of the choices faced by non-nuclear states listed in Table 2.

Row	Policy	utility from acquisition (μ_h)	utility from cooperation (μ_c)	utility from either use (μ_u) or proliferation is high (μ_p)
1	Diplomcay	$< .5$	$< .5$	no
2	none	< 5	$\geq .5$	no
3	Dissuasion and Diplomacy	$\geq .5$	$< .5$	no
4	Dissuasion	$\geq .5$	$\geq .5$	no
5	Dissuasion and Diplomacy	$< .5$	$< .5$	yes
6	Dissuasion	< 5	$\geq .5$	yes
7	Dissuasion and Diplomacy	$\geq .5$	$< .5$	yes
8	Dissuasion	$\geq .5$	$\geq .5$	yes

Table 2

Possible Permutations of the Utility Calculations of Non-Nuclear States

In the first case (row 1), the utility of acquistion is low. Moreover, the utility from both use and proliferation are low as well. Therefore, the likelihood of the non-nuclear state engaging in actions disrupting global nuclear stability is low. However, the utility from cooperation is also low, so the challenge to state k is to engage state i diplomatically to change its utility from cooperation. In the second case (row 2), all parameters remain the same, except that state $i's$ utility from cooperation is relatively high. Thus, state k need do nothing.

In the next two cases, state $i's$ utility from acquistion is high. Therefore, state k must engage in dissuasion, to which it must add diplomacy to increase the utility from cooperation in case 3.

The last four cases are all problematic for state k since state $i's$ relatively high utility from either use or proliferation will lead it to acquire nuclear weapons. Thus, in all four instances, state k must engage in dissuasion; and where the utility from cooperation is low, it must add diplomacy.

7 A Short Concluding Note

We have defined the vocabulary of deterrence and related them to strategies to achieve nuclear stability. The challenge faced by the United States is to achieve nuclear stability by reducing the likelihood of the spread of nuclear weapons (by acquisition or proliferation) as well as their use while increasing international cooperation to achieve these same ends. The challenge is further exacerabted by the reality depicted in Figure 2 that the efforts by states to acquire, proliferate, or use nuclear weapons have interactive affects with those of other states. The challenge facing the U.S. is to reduce any momentum in this direction. However, as we have argued, if the goal is merely to reduce the likelihood of acquisition, proliferation, or use, then dissuasion by the application of some mix of threats of costs and promises of benefits will suffice. However, the goal to achieve cooperation takes the problem set outside of the standard approach. Since our paper has focused largely on the application of hard power, we are less confident of our conclusions, or the adequacy of the analysis of the application of soft power to achieve cooperation. This may well be an area for a substantial amount of research. We urge that these efforts attempt to integrate all aspects of U.S. policy.

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CAUSALITY: A FUTURE DIRECTION FOR MATHEMATICS OF UNCERTAINTY

JOHN N. MORDESON

Department of Mathematics
Creighton University
Omaha, Nebraska 68178 USA
mordes@creighton.edu

TERRY D. CLARK

Department of Political Science
Creighton University
Omaha, Nebraska 68178 USA
tclark@creighton.edu

BEVERLY DOYLE

Department of Education
Creighton University
Omaha, Nebraska 68178 USA
bdoyle@creighton.edu

LYNN OLSON

Department of Education
Creighton University
Omaha, Nebraska 68178 USA
lolson@creighton.edu

In this paper, we present techniques involving mathematics of uncertainty that can be applied to several new research areas. Two of the areas are children with special needs and nuclear stability. Among the techniques given here are the Guiasu method, the analytic hierarchy process, Dempster-Shafer theory, and fuzzy preferences relations. The techniques are used to measure degrees of causality.

Keywords: mathematics of uncertainty; causality, measurement; children with special needs; nuclear stability; Guiasu method; Dempster-Shafer theory; analytic hierarchy process; minimal disagreement approach; data envelopment analysis; fuzzy preference relations.

1. Introduction

Albert Einstein stated that development of Western science is based on two great achievements: the invention of the formal logical system and the discovery of the possibility to determine causal relationships by systematic experiment, Ref.1. In Ref. 1, Judea Pearl states that in the last decade (the 1990s), owing partly to advances in graphical models, causality has undergone a major transformation. Practical problems relying on causal information that long were regarded as unmanageable can now be solved using elementary mathematics. Pearl stresses that basic concepts of probability theory and graph theory is all that is needed for one to begin solving causal problems that are too complex for the unaided intellect.

We support the notion that problems of causality can be studied profitably by the use of mathematics of uncertainty and basic concepts from graph theory. We propose a model consisting of an overarching goal and components making up the overarching goal. We consider the components as nodes with a directed edges emanating from them to the overarching goal. We next consider attributes making up the components as nodes with directed edges emanating from them to the components. Each attribute has at most one edge directed to a particular component, but it may have a directed edge to more than one component. The resulting figure is a directed graph. The importance of each component and each attribute is weighted by means of expert opinion, data, and various mathematical techniques. The weights can be used to determine linear equations, where the overarching goal is the dependent variable and the components the independent variables. Each component can also be the dependent variable for a linear equation of the attributes as the independent variables. By proper substitution, the overarching goal can be written in terms of a linear equation involving the attributes. These linear equations give a measure as to how well the goals are being achieved at a specific time. The independent variables can be considered as causal variables. If directed edges are drawn between attributes to denote a weighted relationship between them, then a resulting mathematical structure is determined and is known as a directed graph or network. This structure can be studied in its own right.

In this paper, we illustrate this model by focusing on several new application areas, e.g., issues involving children with special needs and nuclear stability. Research on the application areas listed below are currently in progress. Students at Creighton University working on these projects are listed next to each application area. In the conclusion, we list the faculty at Creighton and staff from various institutions who are also involved.

Children with Special Needs (Michael Redmond)

There are about 7 million students with disabilities in public schools. Learning disabilities such as dyslexia are the most common, followed by speech or language impairments. Mental retardation and emotional disturbances rank next in frequency, though they show up in much smaller numbers.

The specific focus for deaf and hard of hearing children is on the development of the progression of auditory (listening) skills and the development of understanding and expression of spoken language. The goal of auditory and language instruction is to provide the children with the appropriate language stimulation and practice so the gap will continually close between their chronological age and their language age. The hope is that the children will learn to communicate effectively with their families, their classmates, their hearing peers, and to develop strategies for accessing information to learn in a regular classroom.

Language Environment Analysis (LENA) is a new recording device used in a parent - infant setting with the goals of (1) children making more progress in both receptive and expressive language, (2) professionals having more information, and (3) making parents more effective.

We consider two overarching goals for deaf and hard of hearing prekindergarten children : (1) which tests are the best predictors of success for children when mainstreamed in the public school system and (2) measuring how well children are closing the gap with respect to language growth. For either one of these goals, experts from Omaha Hearing School weighted fifteen test which they thought reflected the importance as indicating the success of obtaining the overarching goal. These weights were converted to numbers from the closed interval [0,1] using the Guiasu method and the analytic hierarchy process. This allowed the overarching goal to be expressed as a linear combination of the tests (the independent variables) with the coefficients of the independent variables the weights from [0,1] and whose sum equals 1.Test scores were coded using numbers from the closed interval [0,1]. These coded numbers can then be substituted into the linear equation to obtain a number from the interval [0,1] that expresses the success of how well the overarching goal is being achieved, Refs. 2, 3.

These ideas and techniques can be applied to measure progress of children with other special needs. Autism is one of five disorders coming under the umbrella of Pervasive Development Disorders, a category of brain disorders that affect the way people speak, socialize, play and react to their environment. There is no cure for autism and no single treatment package for all children with autism. However most professionals agree that early treatment is important and most people with autism respond well to highly structured specialized programs. The techniques described in this paper can be used to measure the success of many of these treatments. Many of these techniques are based on fuzzy set theory, Ref. 4.

Nuclear Stability (Hilary Wething)

An official statement from the Obama Administration stated that the gravest danger to the American people is the threat of a terrorist attack with a nuclear weapon and the spread of nuclear weapons to dangerous regimes. In this paper, we propose the use of fuzzy logic to model the bigger problem of nuclear stability. We base our work on the opinion of experts from government, military, industry, academia, and the literature, Refs. 5-17.

Nuclear stability can be defined as a condition of the international systems in which the actors are disinclined to acquire, proliferate, or use nuclear weapons. Nuclear stability is determined by various factors including: (1) programs to acquire, deploy, proliferate, threaten to use, or use fissile nuclear materials and/or weapons, (2) the security and control regimes of nuclear weapons programs, stockpiles, and fissile materials, (3) state and non-state programs to traffic illegally in nuclear weapons, delivery systems, and/or enabling technologies and materials.

Nuclear stability is made up of four main components: (1) Acquire, (2) Proliferate, (3) Use, and (4) Cooperate. Each component is made up of numerous attributes, namely, (1) enduring rivalries, (2) strategic culture, (3) international openness, (4) political stability, (5) international opinion, (6) credible security guarantee, (7) human capitol, (8) financial resources, (9) economic freedom, (10) globalization, (11) economic stability, (12) distribution networks, (13) delivery systems, (14) C2 (command and control) capability, (15) nuclear infrastructure, (16) threats to state survival, and (17) military stability.

We propose a model with nuclear stability as the overarching goal. We consider the four main components as nodes with a directed edges emanating from them to the nuclear stability node. We next consider attributes making up the components as nodes with directed edges emanating from them to the components. Each attribute has at most one edge directed to a particular component, but it may have a directed edge to more than one component. The resulting figure is a directed graph. The importance of each component and each attribute is weighted by means of expert opinion, data, and various mathematical techniques. The weights will be used to determine linear equations, where nuclear stability is the dependent variable and the four components the independent variables. Each component will also be the dependent variable for a linear equation of the attributes as the independent variables. By proper substitution the overarching goal, nuclear stability, can be written in terms of a linear equation involving the attributes. These linear equations give a measure as to how well the goals are being achieved at a specific time.

Economic Freedom (Hilary Wething)

Economic freedom is the fundamental right of every human to control his or her own labor and property. In an economically free society, individuals are free to work, produce, consume, and invest in any way they please, with that freedom protected by the state and unconstrained by the state. The Wall Street Journal and the Heritage Foundation have tracked the economic freedom of countries for fifteen years, Ref. 18. They have developed an Index of Economic Freedom. The index is based upon analysis of 10 specific components of economic freedom: (1) business freedom, (2) trade freedom, (3) fiscal freedom, (4) government size, (5) monetary freedom, (6) investment freedom, (7) financial freedom, (8) property rights, (9) freedom from corruption, and (10) labor freedom. The 10 composite scores are equally weighted and averaged to get an overall economic freedom score for each country. We propose several weighting systems based on fuzzy mathematics to determine an overall score.

Smart Power (Carly Goodman, Alex Pham)

The Center for Strategic and International Studies formed a bipartisan Commission on Smart Power to develop a vision to guide America's global engagement, Ref. 19. The Commission determined that the United States should concentrate on the following five areas: (1) alliances, partnerships, and institutions, (2) global development, (3) public diplomacy, (4) economic integration, and (5) technology and innovation. These five areas can be considered to be the components making up the overarching goal of improving the image of the United States. Techniques described above can be used to measure the effectiveness of the United States in achieving this goal.

Political Stability (Michael Gibilisco)

Some researchers have defined political stability as a state's ability to avoid revolutionary wars, ethnolinguistic wars, adverse regime changes, genocides and politicides, Refs. 20, 21. With this definition in mind, the overarching goal of political stability can be considered to be made up of the components, (1) economic strength, (2) government legitimacy, (3) institutional stability, and (4) strong external relations. Each of these components has numerous causal factors.

Cooperative Threat Reduction (Alex Pham)

The Defense Threat Reduction Agency program pursues four objectives to reduce the present threat of weapons of mass destruction and guarantee national security, namely, (1) dismantle former Soviet Union (FSU) weapons of mass destruction (WMD) and associated infrastructure, (2) consolidate and secure FSU WMD and related technology and materials, (3) increase transparency and encourage higher standards of conduct, and (4) support defense and military cooperation with the objective of preventing proliferation. Each of these objectives has numerous components, Ref. 22.

Failed States (Alex Pham, Michael Redmond, Michelle Garner)

The Fund for Peace has developed a Failed States Index, Ref. 23. The index is based on 12 factors: (1) demographic pressures, (2) refugees or internally displaced persons, (3) group grievance, (4) human flight, (5) uneven economic development, (6) economic decline, (7) delegitimization of the state, (8) public services, (9) human rights, (10) security apparatus, (11) factionalized elites, and (12) external intervention. The 12 composite scores are equally weighted and averaged to get an overall economic freedom score for each country. We propose several weighting systems based on fuzzy mathematics to determine an overall score.

Economic Stability (Chris Carlson)

Economic stability is made up of the 8 components, (1) low unemployment rate, (2) low inflation rate, (3) little volatility in economic activity, (4) little volatility in exchange rates, (5) little volatility in financial markets, (6) supportive governance practices, (7) increasing productivity, and (8) increasing efficiency, Ref. 24.

Creative Economy (Hilary Wething)

Creative economy or creative industries refers to a collection of interlocking industry sectors, Ref. 25. The UK Government Department for Culture, Media and Sport (DCMS) has produced the following definition of the creative industries: those industries which have their origin in individual creativity, skill and talent and which have a potential for wealth and job creation through the generation and exploitation of intellectual property. The DCMS definition recognizes 11 creative sectors: (1) advertising, (2) architecture, (3) arts and antique markets, (4) crafts, (5) design, (6) designer fashion, (7) film, video, and photography, (8) software, computer games and electronic publishing, (9) music and the visual and performing arts, (10) publishing, (11) television and radio.

Quality of Life (Alex Pham, Chelsea Fischer, Michael Davidson)

The Economist Intelligence Unit, an arm of the Economist Group which publishes The Economist newspaper, periodically collects data from 111 countries and publishes a Quality of Life Index. The nine quality-of-life factors are as follows: (1) material well-being, (2) health, (3) political stability and security, (4) family life, (5) community life, (6) climate and geography, (7) job security, (8) political freedom, (9) gender equality, Ref. 26.

Solar Energy (Michael Davidson)

Renewable energy sources are becoming increasingly popular. They are advantageous over traditional fossil fuels for a number of reasons including independence from unstable, foreign markets and reduction in environmental impact. Photovoltaic energy (PVE), more commonly known as solar power, is an especially promising energy resource. The overarching goal is the proliferation of PVE. The four sub-factors are (1) photovoltaic systems, (2) research and development of PV technologies, (3) economics, and (4) public perception, Ref. 27.

College Freshman Weight Gain (Chelsea Fischer)

As obesity becomes an ever increasing problem across the United States the impact caused by the epidemic is causing major health concerns. Some health conditions that are associated with obesity include cancer, type 2 diabetes, hypertension, stroke, sleep apnea and dyslipidemia. The transition from high school to college is well known for being a time of weight gain in a large portion of the population. This is of particular importance for colleges and universities at which freshmen live on campus and consume food in dining halls. As a result, one area that colleges and universities need to focus on their in their orientation and throughout the first year education is maintenance of a healthy life-style including prevention of weight gain. The prevention of weight gain is the overarching goal and their are various goals that are needed to be addressed in order to reach this overarching goal: (1) food consumed by freshman students, (2) the level of physical activity and exercise, (4) time constraints put on a college student; (5) the level of stress a college student is under, and (6) the social influences. There are a number of factors that have an impact on each goal, Refs. 28-33.

Population Management of Sub-Saharan Africa (Chelsea Mann)

The format includes an organized arrangement of factors that influence population processes, population management policies for the Sub-Saharan region and for each selected country, as well as social, cultural, ethical, and religious issues that shape their current population policies. Three factors making up the overarching goal are (1) processes, (2) management, and (3) social/cultural. Each factor is made up of several subfactors, Refs. 34-38.

Safe Skies (Michelle Garner)

September 11, 2001 opened the world's eyes to a new weapon of mass destruction; commercial airlines. We now live in an age where getting on an airplane requires as much background check and security as getting into the

White House. The overarching goal is keeping commercial airlines from harm. The factors to achieve this goal are (1) layered security, (2) ready response, (3) international cooperation, and (4) maximizing domain awareness, Refs. 39-40.

Denomination Splitting Among Protestants in the United States (Katherine Alexander)

Under the widespread umbrella of the term, Protestantism, is a major international religion with approximately 800 million Protestants worldwide. However, there are a diverse group with over 33,000 denominations in 238 countries. In the United States there are more than 6,000 denominations although these numbers are imprecise due to the ambiguity of the word "denomination." The factors causing splitting are (1) political differences, (2) charismatic issues, and (3) doctrinal differences, Refs.41-42.

Economic Growth in the United States (Chris Carlson)

Economic growth in the United States has fluctuated greatly since the 17th century. Over this time the United States has risen to become the largest economy in the world. Despite the continual growth the economy as been through periods of growth and recession determined by government policy, private sector factors, and communal influences. The factors making up economic growth are (1) government policy, (2) private sector, and (3) community, Refs. 43-50.

Remittances (Jacob Huju)

Remittances are a sum of money (or formerly) a quantity of an item transferred from one place to another. In the context of immigration, remittance usually means money sent to a home country from an immigrant living in a host country. There are three main factors making up remittances: (1) political, (2) social, and (3) economic, Refs. 51-58.

Health Care (Michael Redmond, Chelsea Mann, Jacob Huju)

The results of any adjustment to a complex system are frequently counterintuitive. All U.S. citizens and legal aliens should have access to affordable, minimally adequate health care. Health care reform would involve making heath care available to approximately 45 million additional people. This will involve additional costs. Minimally adequate health care will be based on evidence-based procedures and health economics. The overarching goals are to establish a funds flow model of the current health care delivery system and to determine the most cost-effective method of making affordable, minimally adequate health care available to all U.S. citizens and legal aliens, Ref. 59-68.

Nuclear Deterrence (John Herr, Carly Goodman)

The goal of deterrence is to prevent aggressive action or WMD use by ensuring that, in the mind of the potential adversary, the risks of the action outweigh the benefits, while taking into count the consequences of inaction. The capabilities needed for tailored deterrence involve the full range of military capabilities, presence, and cooperation, as well as diplomatic, informational, and economic instruments, Refs. 69, 70.

Developmental Disorders in Children (Patrick Kilcoyne)

Pervasive development disorders is a group of disorders characterized by delays in the development of multiple basic functions including socialization and communication, Refs. 71, 72. There are three main factors making up developmental disorders: (3) pervasive developmental disorders, (3) neurological disorders, and (4) learning disabilities.

Human Development (Alex Pham)

The Human development Index (HDI) is an index used to rank countries by level of "human development," which usually implies whether a country is developed, developing, or underdeveloped, Ref. 73. The HDI combines three dimensions: (1) life expectancy at birth, is an index of population health and longevity, (2) knowledge and education, as measured by the adult literacy ratio (with two-thirds weighting) and the combined primary, secondary, and tertiary gross enrollment ratio (with one-third weighting), (3) standard of living, as measured by the natural logarithm of gross domestic product per capita at purchasing power utility.

Child Development (Alex Pham)

The Save the Children Fund (SCF) is a charity registered in England and Whales, Ref. 74. Its purpose is to change the plight of children that are denied proper health care, food, education, and protection. SCI highlights the following three areas: (1) nutrition, (2) equitable development, and (3) women's education and empowerment.

We discuss in the following new methods that can be used to determine the coefficients of the independent variables of the previously mentioned linear equations.

2. Guiasu's Method

Guiasu's method describes the process of reaching a verdict by a probabilistic weighting of the expert opinions, which permits us to determine a numerical relationship between a goal and the factors making up the goal. The classical rules from decision theory proposed by Hooper, Dempster, Bayes, and Jeffrey are special cases of Guiasu's weighting process (see Ref. 75, p. 167). Guiasu addresses differences in expert opinion over the likelihood (probability) of something being true as well as the credibility of the claims (our belief in its truth).

A body of information induces a credibility (or probability) distribution m on $\mathcal{P}(X)$, the set of all subsets of X. That is, m is a function of $\mathcal{P}(X)$ into the closed interval $[0,1]$, written $m : \mathcal{P}(X) \to [0,1]$, such that $m(A) \geq 0 \forall A \in \mathcal{P}(X)$ and $\sum_{A \subseteq X} m(A) = 1$. The class of focal subsets of X corresponding to m is denoted by $\mathcal{F}(X;m) = \{A \mid A \subseteq X, m(A) > 0\}$. A pair of dependent bodies of information, say i and j, induce a joint credibility distribution, namely $m_{ij} : \mathcal{P}(X) \times \mathcal{P}(X) \to [0,1]$ such that $m_{ij}(A,B) \geq 0$ and

$$\sum_{A \subseteq X} \sum_{B \subseteq X} m_{ij}(A,B) = 1.$$

If the bodies of information are independent, then $m_{ij} = m_i m_j$. The corresponding class of focal pairs of subsets is $\mathcal{F}(X,X;m_{ij}) = \{(A,B) \mid A,B \subseteq$

$X, m_{ij}(A,B) > 0\}$. The weights corresponding to the body of information for which m is the credibility distribution are $w(\cdot \mid \cdot) : \mathcal{P}(X) \times \mathcal{F}(X;m) \to [0,\infty)$. The weighted body of information provides the new credibility distribution on $\mathcal{P}(X)$ given by

$$\mu(C) = \sum_{A \in F(X;m)} w(C \mid A) m(A).$$

We can generalize this procedure to formulate the way weights $w_{ij}(\cdot \mid \cdot, \cdot)$ are assigned to a mixed body of information inducing a joint credibility distribution induced on $\mathcal{P}(X)$. The weighted (i,j)-th body of information is

$$\mu_{ij}(C) = \sum_{(A,B) \in \mathcal{F}(X,X;m_{ij})} w_{ij}(C \mid A,B) m_{ij}(A,B), C \in \mathcal{P}(X),$$

where $w_{ij}(C \mid A,B)$ is the weight of the subset C given $(A,B) \in \mathcal{F}(X,X;m_{ij})$. This can, of course, be extended to any number of dimensions.

3. Belief and Plausibility Functions

Dempster Shafer theory was developed to deal with subjective probabilities in which probability measures one's degree of belief in a statement or hypothesis (that is, the credibility of the statement), Ref. 76. It is based on two dual nonadditive measures, namely, belief measures and plausibility measures. These measures can be defined in terms of a function $m : \mathcal{P}(X) \to [0,1]$ such that $m(\emptyset) = 0$, where $\emptyset$ denotes the empty set, and $\sum_{A \in \mathcal{P}(X)} m(A) = 1$. The function m is called a basic probability assignment. For all $A \in \mathcal{P}(X)$, $m(A)$ is the proportion to which all available and relevant evidence supports the claim that a particular element of X belongs to A. The number $m(A)$ pertains only to the set A. The value $m(B)$ for a subset B of A cannot be determined from A. For example, it is not required that $m(B) \leq m(A)$. It is also not required that $m(X) = 1$ or that there be a relation between $m(A)$ and $m(\overline{A})$. A belief function is defined in terms of m in the following manner: $Bel : \mathcal{P}(X) \to [0,1]$ such that $\forall B \in \mathcal{P}(X), Bel(B) = \sum_{A|A \subseteq B} m(A)$. The plausibility function is defined by $Pl(B) = \sum_{A|A \cap B \neq \emptyset} m(A)$. For all $A \in \mathcal{P}(X), Bel(A)$ is interpreted as the degree of belief based on available evidence that a given element of X belongs to A.

Evidence obtained in the same context from two independent sources and expressed by basic probability assignments m_1 and m_2 on $\mathcal{P}(X)$ can be combined to obtain a joint basic probability assignment m_{12}. We sketch the standard way of combining them. Let $K = \sum_{A \cap B = \emptyset} m_1(A) m_2(B)$. Suppose $K < 1$. Define $m_{12} : \mathcal{P}(X) \to [0,1]$ by $\forall C \in \mathcal{P}(X)$,

$$m_{12}(C) = \sum_{A \cap B = C} m_1(A) m_2(B)/[1-K],$$

if $C \neq \emptyset$ and $m_{12}(\emptyset) = 0$. Then the above formula is referred to as Dempster's rule of combination.

If the basic probability assignments are determined by expert opinion, say of factors making up a goal, then the belief of a subset of the set of all factors can be interpreted as the degree of importance of that subset in determining the overarching goal. Also, if the basic probability assignments are positive only on singleton sets, say for example single factors, then the combination of these basic probability assignments into a single basic probability assignment can be used as coefficients for the factors of the linear equation of the overarching goal expressed in terms of its factors.

Yen's method, Ref. 77, is a generalization of the Dempster-Shafer theory to allow for fuzzy subsets. It addresses the issue of managing imprecise and vague information in evidential reasoning by combining the Dempster-Shafer theory with fuzzy set theory. Several researchers have extended the Dempster-Shafer theory to deal with vague information, but their extensions did not preserve an important principle that the belief and the plausibility measures are lower and upper probabilities. Yen's method preserves this principle by employing normalized fuzzy measures of credibility. It also preserves the property that the belief of a (fuzzy) subset is the difference of one and the plausibility of the subset's complement. Further, it is more responsive to a change to a focal element's membership function than some approaches.

Yen's method involves various measures of subsethood and extends Dempster-Shafer's belief function by defining a measure of inclusion $I(A, B)$, the degree to which the fuzzy subset A is included in the fuzzy subset B by using the following formula,

$$Bel(B) = \sum_{A \in \mathcal{FP}(X)} I(A, B)m(A),$$

where $m : \mathcal{FP}(X) \to [0, 1]$ is such that $\sum_{A \in \mathcal{FP}(X)} m(A) = 1$ and where $\mathcal{FP}(X)$ denotes the set of all fuzzy subsets of X.

Let X be a nonempty set and A and B be fuzzy subsets of X. Let $\alpha \in [0, 1]$ and $A^\alpha = \{x \in X \mid A(x) \geq \alpha\}$. Then A^α is called an α-cut of A. Define $\overline{A} : X \to [0, 1]$ by $\forall x \in X, \overline{A}(x) = 1 - A(x)$. Then $\overline{A}$ is known as the standard complement of A. Let $I : \mathcal{FP}(X) \times \mathcal{FP}(X) \to [0, 1]$. Fuzzy set theory uses minimum to model intersection, represented with $\wedge$, and maximum to model union, represented by $\vee$.

There are a number of alternatives to Yen's method. They use measures of fuzzy subsethood. These approaches combine Dempster-Shafer theory and fuzzy set theory. The following are several measures of inclusion (the degree to which a fuzzy subset is a subset of another) (see Refs. 78-89): $\forall A, B \in \mathcal{FP}(X)$:

Bandler and Kohout:

$$I_{BK}(A, B) = \wedge\{\overline{A}(x) \vee B(x) \mid x \in X\},$$

Sanchez:

$$I_S(A, B) = \frac{\sum_{x \in X}(A(x) \wedge B(x))}{\sum_{x \in X} A(x),}$$

Wierman:

$$I_W(A,B) = \int_0^1 \chi_{\subseteq}(A^\alpha, B^\alpha)d\alpha,$$

where $\chi_{\subseteq}(A^\alpha, B^\alpha) = 1$ if $A^\alpha \subseteq B^\alpha$ and 0 otherwise.

4. Analytic Hierarchy Process

The Analytic Hierarchy process (AHP) provides the objective mathematics to express the subjective and personal preferences of an individual or a group in making a decision, Ref. 90. With the AHP and its generalization, the Analytic Network Process (ANP), one constructs hierarchies or feedback networks, then makes judgements or performs measurements on pairs of elements with respect to a controlling element to derive ratio scales that are then synthesized throughout the structure to select the best alternative.

Fundamentally, the AHP works by developing priorities for alternatives and the criteria used to judge the alternatives.

AHP is a systematic method for comparing a list of alternatives or objectives. Assume that a set of objectives has been established and that we wish to establish a normalized set of weights to be used when comparing alternatives using these objectives. We form a pairwise comparison matrix $A = [a_{ij}]$, where the number in the i-th row and j-th column gives the relative importance of objective O_i as compared with objective O_j. One may use a $1-9$ scale, where

$$a_{ij} = \begin{cases} 1 \text{if } O_i \text{ and } O_j \text{ are of equal importance}, \\ 3 \text{ if } O_i \text{ is weakly more important than } O_j, \\ 5 \text{ if } O_i \text{ is strongly more important than } O_j, \\ 7 \text{ if } O_i \text{ is very strongly more important then } O_j, \\ 9 \text{ if } O_i \text{ is absolutely more important than } O_j, \end{cases}$$

and

$$a_{ij} = \begin{cases} 1 \text{if } O_j \text{ and } O_i \text{ are of equal importance}, \\ 1/3 \text{ if } O_j \text{ is weakly more important than } O_i, \\ 1/5 \text{ if } O_j \text{ is strongly more important than } O_i, \\ 1/7 \text{ if } O_j \text{ is very strongly more important then } O_i, \\ 1/9 \text{ if } O_j \text{ is absolutely more important than } O_i, \end{cases}$$

The next step is to normalize the columns of A by computing the sum of each column and then divide each column by the corresponding sum. Ideally the normalized columns would all be identical if the pairwise comparisons were consistent. In practice, one can compute a consistency measure using the eigenvalues of the normalized matrix. The next step is to compute the average value of each row and use these values as the weights in what's called the Objective Hierarchy. These weights sum to 1. These weights would be used in summing the measures as required in the evaluation of the Objective Hierarchy.

The use of a $1-9$ scale is not the only approach. Experts may assign weights $w_i i = 1, ..., n$, of importance to the n priorities. Emphasis on consistency leads to the eigenvalue formulation $Aw = nw$, where n is the number of priorities and $w = (w_1, ..., w_n)$ is the vector consisting of the weights for the priorities.

Then the matrix of ratio comparisons $A = [w_i/w_j]$ is formed. Thus we have $Aw = nw$.

If a_{ij} represents the importance of alternative i over alternative j and a_{jk} represents the importance of alternative j over alternative k, the importance of alternative i over alternative k, must equal $a_{ij}a_{jk} = a_{ik}$ for the judgements to be consistent.

5. Data Envelopment Analysis

Data Envelopment Analysis (DEA) is a data oriented approach for evaluating the performance of a set of peer entities, Refs. 91, 92. These entities are called Decision Making Units (DMUs). They convert multiple inputs into multiple outputs. There is a large variety of applications of DEA for use in evaluating the performance of many different types of entities. DEA requires very few assumptions. Consequently, possibilities have been opened for use in cases which have been resistant to other approaches due to the complex or unknown nature of the relations between the multiple inputs and multiple outputs in DMUs.

DEA is a fractional programming problem in linear programming methodology used when a direct causal relationship between inputs and outputs cannot be claimed. DEA measures the relative efficiency of these DMUs by finding the maximum of the ratio of the weighted outputs to inputs under the condition that the ratios of all the DMUs must be less than or equal to 1. Efficiency is described as the weighted sum of outputs to the weighted sum of inputs where the weights structure is figured out by mathematical programming. This weighting of "relative efficiency' is determined objectively to obtain a scalar measure of efficiency in any case. The DMU's are inefficiencies in inputs or outputs with "slack variables," and a DMU can be rated efficient if the maximum ratio is equal to 1 (100% return to scale) and the slack variables equal zero. There are four basic DEA models:

(1) the CCR ratio model,
(2) the BCC model,
(3) the Multiplicative model,
(4) the additive and the extended additive model.

The CCR model results in a piecewise linear, constant returns-to-scale envelopment surface. The BCC and Additive models yield in a piecewise linear, variable returns-to-scale envelopment surface. The Multiplicative models yield piecewise log-linear envelopment surfaces.

We sketch give a brief sketch of the "ratio-form" of DEA. Assume there are n DMUs to be evaluated. Each DMU consumes varying amounts of m different inputs to produce s different outputs. Specifically, DMU_j consumes amount x_{ij} of input i a produces amount y_{rj} of output r. We assume that $x_{ij} \geq 0$ and $y_{rj} \geq 0$. We also assume that each DMU has at least one positive input and one positive output value. The ratio of outputs to inputs is used to measure the relative efficiency of the DMU to be evaluated, say DMU_o, relative to the ratios of the DMU_j, $j = 1, 2, ..., n$. The CCR construction can be interpreted as the reduction of the multiple-output/multiple-input situation (for each DMU) to that of a single 'virtual' output and 'virtual' input. For a particular DMU the ratio of this single virtual output to single input provides a measure of efficiency that is a function of the multipliers. In mathematical programming terms, this ratio which is to be maximized, forms the objective function for the particular DMU being evaluated. That is,

$$\max h_o(\mathbf{u},\mathbf{v}) = \sum_{r=1}^{s} u_r y_{ro} / \sum_{i=1}^{m} v_i x_{io},$$

where $\mathbf{u} = (u_1, ...u_s)$ and $\mathbf{v} = (v_1, ..., v_m)$ and where the u_r and v_i are the variables and the y_{ro} and x_{io} are the observed output and input values, respectively, of the DMU_o to be evaluated. In order keep the solution from being unbounded, a set of normalizing constraints for each DMU is specified so that the ratio must be less than or equal to 1. The mathematical programming problem then becomes

$$\begin{aligned} \max h_o(\mathbf{u},\mathbf{v}) &= \sum_{r=1}^{s} u_r y_{ro} / \sum_{i=1}^{m} v_i x_{io}, \\ &\ \text{subject to} \\ \sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij} &\leq 1 \text{ for } j = 1, ..., n, \\ u_r, v_i &\geq 0 \text{ for } r = 1, ..., s; i = 1, ..., m. \end{aligned}$$

6. Minimum Disagreement Approach

The Minimum Disagreement Approach (MDA) doesn't find a single best explanation, but a number of good explanations that satisfy pairwise independence, Ref. 93. The best hypothesis from a fixed set of alternative hypotheses is selected to determine a causal relationship. This is most frequently used with "noisy data," a term used in information extraction analysis.

We assume we have an overarching goal G supported by goals $G_i, i = 1, ..., n$, for which we wish to measure using k predetermined factors. Let Z_{ij} be the score of factor j for $Gi, j = 1, ..., k$. We impose the following rules for each G_i:

We assume the overall score G_i is a linear function of the Z_{ij}, i.e.,

$$G_i = \sum_{j=1}^{k} w_j Z_{ij},$$

where G_i is the overall score of goal i and w_j is the importance weight of factor $j, j = 1, ..., k$.

We require

$$\sum_{j=1}^{k} w_j = 1.$$

In order to insure each factor has an impact on G_i, we impose bounds on each w_j,i.e.,

$$b_j \geq w_j \geq c_j,$$

where the b_j and c_j are predetermined by the evaluator,$j = 1, ..., k$.

The problem becomes given the factor rating Z_{ij} how is a ranking of the G_i determined that minimizes the conflicts and disagreements among the G_i. For this model, there are two stages. First the set of weights for each G_i is calculated giving each G_i the best ranking. Second, the model aggregates ranks from all goals to resolve conflicts with minimum disagreement.

7. Fuzzy Preference Relations

Once the experts have provided their weights as to the importance of the factors, there are several techniques, Refs. 94-97, that can be applied to these weights to determine fuzzy preference relations on X, the set of factors. Let ρ_k denote the fuzzy preference relation for expert $k, k = 1, ..., n$. There are many techniques which can be used to aggregate these fuzzy preference relations into a single fuzzy preference relation ρ. Each of these fuzzy preference relations can be represented by an $m \times m$-matrix. Let R_k denote the matrix $[r_{ij}^k]$ associated with $\rho_k, k = 1, ..., m$ and R denote the matrix $[r_{ij}]$ associated with ρ. We propose that these matrices can be used to employ fuzzy linguistic quantifiers to represent the notion of a fuzzy majority. Some fuzzy logic-based calculi of linguistically quantified propositions have been proposed which can make it possible to handle fuzzy linguistic quantifiers. These calculi have been applied in order to introduce a fuzzy majority (represented by a fuzzy linguistic quantifier) into group decision making and consensus formation models and also an implemented Decision Support System for consensus reaching.

An example of a *linguistically quantified proposition* is "most experts are convinced" and may generally written as Qy's are F, where Q is a linguistic quantifier, e.g., most, Y is a set of experts with $y \in Y$, and F is a *property*, e.g., convinced. the problem becomes to find the truth of such statements. Two basic calculi may be employed Ref. 98 and Refs. 99, 100. We present a method proposed by Zadeh. In Ref. 99, a fuzzy linguistic quantifier Q is assumed to be a fuzzy subset. For instance, Q = 'most' may be given as

$$\mu_Q(x) = \begin{cases} 1 \text{ for } x \geq .8, \\ 2x - .6 \text{ for } .3 < x < .8, \\ 0 \text{ for } x \leq .3. \end{cases}$$

We propose the use of a fuzzy majority represented by a linguistic quantifier to determine the most important factors using the R_k and R. The *core* is an appealing concept to use. The core is defined as a set, C, of undominated factors, i.e., those not defeated in pairwise comparisons by a strict majority $r \leq n$. That is, $C = \{Fj | \nexists F_i \text{ such that } r_{ij}^k > .5 \text{ for at least } r \text{ experts}\}$. The concept of a core and other concepts to follow in this section can be generalized by replacing .5 by some $\alpha \in (0, 1)$.

Let

$$h_{ij}^k = \begin{cases} 1 \text{ if } r_{ij}^k < .5, \\ 0 \text{ otherwise.} \end{cases}$$

where $i, j = 1, ..., m$ and $j = 1, ..., n$. thus $h_{ij}^k = 1$ if factor Fj defeats Fi and $h_{ij}^k = 0$ otherwise. Let H_k denote the matrix $[h_{ij}^k], k = 1, ..., n$. Then

$$h_j^k = \frac{1}{m-1} \sum_{i=1, i \neq j}^{m} h_{ij}^k$$

is the extent to which expert k is not against factor Fj. Thus $h_j = \frac{1}{n} \sum_{k=1}^{n} h_j^k$ is the extent to which all the experts are not against Fj and $\nu_Q^j = \mu_Q(h_j)$ is the extent Q individuals are not against Fj. The fuzzy Q-core is defined to be the fuzzy subset

$$C_Q = \nu_Q^1 / F1 + ... + \nu_Q^m / Fm,$$

i.e., a fuzzy subset of factors that are not defeated by Q experts.

Now let R denote the matrix $[r_{ij}]$, where

$$r_{ij} = \begin{cases} \frac{1}{m} \sum_{k=1}^{m} a_{ij}^k \text{ if } i \neq j, \\ 0 \text{ otherwise,} \end{cases}$$

where

$$a_{ij}^k = \begin{cases} 1 \text{ if } r_{ij}^k > .5, \\ 0 \text{ otherwise.} \end{cases}$$

Let

$$g_{ij} = \begin{cases} 1 \text{ if } r_{ij} > .5, \\ 0 \text{ otherwise} \end{cases}$$

which expresses whether or not s_i defeats s_j. Then

$$g_i = \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} g_{ij}$$

is the mean degree to which option s_i is preferred to all other options. Let $z_{Qi} = \mu_Q(g_i)$, the extent to which s_i is preferred to Q other options. The fuzzy Q-consensus winner is defined to be the fuzzy subset W_Q of the set of options by

$$W_Q = z_Q^1 / s_1 + ... + z_Q^n / s_n.$$

W_Q represents the fuzzy subset of options that preferred to Q other options.

Other concepts that can be defined and used are the *degree of consensus*, *relevance of options*, *relevance of a pair of options*, and the *importance of experts.* The *degree of agreement* of experts can also be measured.

The technique of preference modelling from outranking relations can be employed. Let $S, I, \succ, R$ and $\tilde{}$ be relations on X, where $x, y \in X$,

S : **Outranking relation**. The statement xSy means "x is not worse than y" and reflects the presence of arguments strong enough to support this assertion.

I : **Indifference relation**. The statement xIy means "x and y are indifferent, i.e., roughly equal, and reflects the presence of arguments strong enough to support both the assertions xSy and ySx.

$\succ$: **Preference relation** The statement $x \succ y$ means "x is preferred to y" and both the presence of arguments strong enough to support the assertion xSy and the absence of similar arguments to support the assertion ySx.

R : **Incomparability relation** The statement xRy means x and y are incomparable" and represents the absence of arguments strong enough to support at least one of the assertions xSy or ySx.

$\tilde{}$: **Non-preference relation The statement** $x\tilde{}y$ means "we cannot discriminate between and y, x and t are either incomparable or indifferent": it reflects the absence of arguments strong enough to support at least one of the two assertions $x \succ y$ or $y \succ x$.

The relations $S, I, \succ, R$, and $\tilde{}$ can be fuzzified in a natural way as follows: Let (N, T, V) be a de Morgan triple such that $\forall x \in [0,1], T(x, N(x)) = 0$, where N is a negation operator, T is a t-norm, and V is a s-conorm. Let $I, \succ_{N,T}, R_N$, and $\tilde{}_{N,T}$ be defined in terms of S as follows: $\forall x, y \in X$,

Indifference Index: $i(S)(x,y) = S(x,y) \wedge S(y,x)$;

Incomparability Index: $R_N(S)(x,y) = N(S(x,y)) \wedge N(S(y,x))$;

Preference Index: $\succ_{N,T} (E)(x,y) = T[S(x,y), N(S(x,y))]$;

Non-preference Index: $\tilde{}_{N,T}(S)(x,y) = (N(\succ_{N,T} (S)(x,y)) \wedge (N(\succ_{N,T} (S)(y,x))$.

8. Strengthening and Weakening Nodes in a Digraph

In a digraph, there are nodes which may be exercising a disruptive of divisive influence while others are helpful. The structure may be built from many different kinds of relationships, such as sociometric choice, communication, or power. It is useful to have a way of characterizing, in digraph terms, the amount to which nodes contribute to the connectedness of the digraph, Refs. 101, 102.

9. Maximal Flow

In our digraph, there is a natural flow from the attributes to the overarching goal. Assuming that our digraph is a transport network, a simple, directed graph with a vertex, the source, with no incoming edges, a vertex, the sink, with no outgoing edges, and weights on the edges, capacities, we can consider the standard problem of maximal flow.

One can consider the problem of maximum flow with fuzzy network arc capacities. There are results which can be considered equivalent to the classic Ford and Fulkerson theorem concerning the minimum cut and maximum flow value. One may also consider real-valued flows in a network with fuzzy capacity constraints. This approach uses the notion of a fuzzy number since the arc capacities will be fuzzy numbers.

10. Conclusions

In this paper, we present mathematical techniques involving mathematics of uncertainty that can be applied to a wide variety of new research areas. Those involved in these research projects are: Dr. M. J. Wierman (Computer Science), Dr. Fong (Statistics), Drs. M. and D. Mallenby (Economics, Statistics), Ms. M. Kelly, Ms. L. Burton, Ms. B. Walker, Ms. A. Bell (Omaha Hearing School), Sister M. Faltus and Ms. Cathy Hirchert (Madonna School).

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Characteristics of Strategy-Proof Fuzzy Choice*

Michael Gibilisco†

11/28/10

Abstract

This paper considers the strategic manipulation of fuzzy choice functions where both individuals and groups can choose alternatives to various degrees. Past efforts to model fuzzy choice and strategic manipulation have allowed individual preferences to be fuzzy but still required groups to select only one alternative (e.g. Abdelaziz, José and Meddeb [1]; Côrte-Real [14]). Under this new framework, I find, with very minimal assumptions on fuzzy preferences, strategy-proof fuzzy choice functions satisfy fuzzy versions of peak-only, weak Paretianism and monotonicity. In addition, the only type of strategy-proof fuzzy choice function corresponds to the traditional augmented median rule. Further, I illustrate the implications this framework in the spatial model. These results are relevant to the manipulation literature, which remains divided as to whether choice functions can be both non-manipulable and non-dictatorial when restricting individual preferences to a single-peaked domain (e.g. Mackie [24]; Penn, Patty and Gailmard [30]). In this context, the paper suggests that social choice can be both strategy-proof and non-dictatorial if alternatives are chosen to various degrees.

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†Department of Political Science, Creighton University: michaelgibilisco@creighton.edu

1 Strategy-Proofness and Fuzzy Sets

A collective choice function over three or more alternatives that does not incentivize individuals who misrepresent their true preferences must be dictatorial [19, 34]. It then follows that voters in collective choice institutions will manipulate the voting procedure to obtain a more preferred social outcome by misrepresenting their true preferences. Social choice scholars have tried to avoid this conclusion of the Gibbard-Satterthwaite (G-S in what follows) theorem by relaxing several of its original assumptions. One approach restricts the domain of individual preferences to single-peaked profiles and finds that the augmented median rule emerges as a non-manipulable and non-dictatorial choice function [5, 9, 26, ?]. While some scholars (e.g. Dryzek and List [16]; Mackie [24]) hold that this restriction voids the results of the G-S theorem, Penn, Gailmard and Patty [30] extend the G-S results to a general case by demonstrating that even though individuals may possess single-peaked preferences, there exists opportunities to manipulate the social choice when individuals report false preferences that violate the natural ordering of the alternatives.

What has remained absent from this debate has been the effects of fuzzy preferences and fuzzy social choice on G-S's conclusion. In the fuzzy framework, individauls can prefer one alternative over another to a certain degree instead of only possessing strict preference or indifference between the two [4, 29]. The addition of fuzzy preferences then requires the specification of a fuzzy social choice function that selects some type of outcome. Although past efforts have explored situations where actors have fuzzy preferences but, as group, must make exact choices, i.e. where society selects one alternative unequivocally and have only confirmed the G-S conclusion [1, 14, 37], the strategic manipulation of truly fuzzy social choice functions, where society chooses alternatives to certain degrees, has yet to be considered.

The purpose of this paper is to address this lacuna in the manipulation literature. To do so, it integrates two recent developments in fuzzy social choice theory. First, it considers the specific fuzzy choice functions proposed by Dasgupta and Deb [15] and Banerjee [4]. These choice functions allow a set of individuals to select more than one alternative with varying degrees of choice. While there exist a significant literature investigating the manipulation of social choice correspondences that allow multiple alternatives to be selected (e.g. Kelly [22] and Barberà, Dutta and Sen [6]), the degree to which the group actors select the alternatives has not been allowed to vary. Second, the paper uses the fuzzy preference framework proposed by Nurmi [28], which employs preference functions instead of preference relations to describe individual preferences. A preference function, unlike a preference relation, accounts for an individual's preference over the set of alternatives instead of over the set of ordered pairs of alternatives. This approach not only allows for a direct comparison between individual preferences and the social choice, but it is also more conducive to empricial testing and application because it does not require an individual to immediately specify his or her preference comparing every alternative to another [8, 12, 13].

Under this setup, I characterize fuzzy choice functions and demonstrate that strategy-proofness implies fuzzy versions of weak Paretianism, peak-only and monotonicity. Further, strategy-proofness is necessary and sufficient for the augmented median voter rule. These results suggest that when the social choice is allowed to be fuzzy, there exists a class of non-manipulable choice functions, which do not replace any type of transitivity or single-peaked restriction on individual preferences. In addition, I illustrate the model using fuzzy spatial preferences. The paper proceeds as follows. Section two briefly reviews the literature discussing fuzzy manipulation. Section three presents the main concepts and definition. Section four details the main findings of the paper. Finally, section five offers a discussion and a critique of the social choice model in context of the spatial model, and section 6 concludes the paper.

2 Previous Attempts at Fuzzy Choice

Fuzzy Choice and Manipulation

Most efforts incorporating fuzzy mathematics into social choice functions start with a fuzzy preference relation, which is a function $\rho : X \times X \rightarrow [0,1]$ where X is the set of alternatives. In words, $\rho(x, y)$ refers to the degree to which x is at least as good as y. If $\rho(x, y) = 1$, then x is said to be definitely as least as good as y; if $\rho(x, y) = 0$, then x is said to be definitely not as least as good as y. When $\rho(x, y) \in (0, 1)$, the preference for x over y is said to be vague or ambiguous.[1] For a set of n actors,N, previous definitions of fuzy choice functions associates an n-tuple of fuzzy preference relations with one alternative in X [1, 14, 29, 37].

Because a fuzzy preference relation is not directly comparable to a subset of alternatives, scholars have considered various mechanisms to aggregate individual preference relations into a social choice. Intitial studies assumed that individuals possess fuzzy preferences but must make "crisp" individual choices over the set of alternatives, and the choice function associates a set of alternatives to these crisp choices [14, 29]. Such situations arise when actors, who possess fuzzy preference relations must vote, "yes" or "no" for an amendment or select only one candidate among many. Later research aggregates a collection of individual preference relations into a social preference relation and then associates an alternative with the fuzzy social preference relation [1]. The following example illustrates the difference between the two approaches.

Example. Let $X = \{a, b\}$ and $N = \{1, 2, 3\}$. Suppose $\rho_1(a, b) = .4, \rho_2(a, b) = .4, \text{and} \rho_3(a, b) = .9$. In words, ρ_i is the fuzzy individual preference relation associated with $i \in N$. Futhermore, suppose reciprocity in preferences, and accordingly, $\rho_i(b, a) = 1 - \rho_i(a, b)$.

Orlovsky Rule. The Orlovsky [29] rule demonstrates the first approach to fuzzy choice, where each actor must make a crisp decision. The Orlovsky rule – or a variation of it – is a function that maps a set of two alternatives into the set $\{0, 1\}$, formally, $IC_i : X \rightarrow \{0, 1\}$.

[1] For a more thorough review fuzzy preferences and how they relate to tradition preferences see Orlovsky [29], Dutta [18], Richardson [33] and Llamazares [23].

More specifically, let $x, y \in X$ and the Orlovsky rule be defined as follows:

$$IC_i(\rho_i)(x) = \begin{cases} 1 & \text{if } R_i(x,y) > R_i(y,x) \\ 0 & \text{else} \end{cases}$$

In words, i votes for, or chooses, x over y if and only if $IC_i(\rho_i)(x) = 1$ and $IC_i(\rho_i(y) = 0$. Considering the above example, $IC(\rho)(a) = (IC_1(\rho_1)(a), IC_2(\rho_2)(a), IC_3(\rho_3)(a)) = (0, 0, 1)$, $IC(\rho)(b) = (IC_1(\rho_1)(b), IC_2(\rho_2)(b), IC_3(\rho_3)(b)) = (1, 1, 0)$, because actors 1 and 2 choose b and actor 1 chooses a. We can tally the votes in any number of ways, but under majority rule b is the outcome.

Mean Aggregation Rule. The mean aggregation rule is a fuzzy aggregation rule and demonstrates the second type of fuzzy choice function, where actors need not make crisp decisions. The mean aggregation rule is defined as follows:

$$\rho_S(x,y) = \frac{1}{n}\sum_{i=1}^{n} \rho_i(x,y)$$

Using the mean aggregation rule, we can specify the social fuzzy preference relation, which is $\rho_S(a,b) = .567$ and $\rho_S(b,a) = .433$. When we use the Orlovsky rule on ρ_S, the social choice becomes a because $\rho_S(a,b) > \rho_S(b,a)$.

In both conceptualizations, the group only selects one, exact alternative even though the choice functions are said to be fuzzy. Further, they both return identical results to the G-S theorem where a choice function is non-manipulable if and only if it is dictatorial[1, 37]. Côrte-Real [14] demonstrates the Orlovsky rule is strategy proof but considers only two alternatives. Nonetheless, new results may be obtained when considering the fuzzy choice functions proposed by Dasgupta and Deb [15] and Banerjee [4]. Under their framework, choice is represented as a fuzzy subset of the set of alternatives, i.e. $\beta : X \to [0,1]$. For any alternative $x \in X$, $\beta(x)$ denotes the degree to which x is chosen. While the conceptualization of fuzzy choice has received a great deal of attentiaion in revealed preference theory[20, 21], the possiblity of manipulating these types of choice functions has yet to be considered.

Individual Preferences and Manipulation

Before presenting a model of fuzzy choice, a brief discussion on the types of fuzzy individual preferences is needed. Informally, a choice function is manipulable by an actor if the actor can unilaterally change the social choice in her favor by submitting an insincere or false preference. To address this formally, it follows that there exist some mechanism to compare the social choice with an individual's preferences. In the exact case, each individual possesses a transitive ranking of the alternatives and a choice is manipulable if there exists an individual who can move the social choice further up her ranking. In the fuzzy case, this mechanism relating individual preferences to the social choice is more complicated. When comparing individual fuzzy preference relations and an exact social choice, Abdelaziz, Figueira and Meddeb [1] utilize four different procedures determining whether an individual prefers one alternative over another, hence four definitions of manipulability. Perote-Peña and Piggins [31] offer one solution to the problem by considering the manipulation of fuzzy aggregation rules, where an n-tuple of fuzzy preference relations are aggregated into a single social preference relation. While this setup has only confirmed the G-S theorem in the fuzzy framework (see Duddy, Perote-Peña and Piggins [17] for a general proof), modeling individual preferences as fuzzy subsets of the set of alternatives rather than fuzzy relations simplifies the forthcoming analysis.

Further, representing individual preferences in this manner is not completely divoriced from preference relations. Dasgupta and Deb [15] and Georgecuscu [21] illustrate how fuzzy subsets can be related to fuzzy preference relations using concepts similar to R-maximality and R-greatness revealed preference theory (see Suzumura [36] and Sen [35] for reference). In addition, Clark, Larson, Mordeson, Potter and Wierman [11] discuss several substantive interpretations of fuzzy subsets of the set of alternatives as representations of individual preference. For example, let β be a fuzzy subset of X and $x \in X$. When $\beta(x)$ refers to the degree to which x is ideal, actors are uncertain how ideal each alternative is; however, the they are quite certain whether x is better, or preferred to, another alternative $y \in X$.

3 Fuzzifying Collective Choice

This section details the fuzzy choice framework and introduces the concepts of strategy-proofness used in the model. Let X be a set of alternatives either finite or infinite. A fuzzy subset of X, f, is a function $f: X \to [0,1]$. Let $\mathcal{F}(X)$ denote all possible fuzzy subsets of X. Let N be a finite set of individuals, where $N = \{1, 2, \ldots, n\}$ and $n \geq 2$.

Definition 1. (*Fuzzy preference function*). An individual fuzzy preference function on X associated with $i \in N$ is a function $\sigma_i: X \to [0,1]$.

Obviously, $\sigma_i \in \mathcal{F}(X)$. Suppose $x \in X$. Then the literature, $\sigma_i(x)$ provides two substantive interpretations of a fuzzy preference function. First, $\sigma_i(x)$ can refer to the degree to which $i \in N$ views x as ideal, where $\sigma_i(x) = 0$ means i believes x is abhorrent and $\sigma_i(x) = 1$ means i believes x is ideal [28, 11, **?**]. Second, $\sigma_i(x)$ can also be interpretted as the degree to which i chooses the alternative x [4, 20, 21]. In this paper, $\sigma_i(x)$ is often refered as the choice instensity of individual i for alternative x. The *profile* of all individual fuzzy preference functions can be written as $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. $\sigma(x) = (\sigma_1(x), \sigma_2(x), \ldots, \sigma_n(x))$ denotes the restriction of σ to x. A fuzzy choice function then associates a fuzzy subet of X to a profile of individual fuzzy preference functions.

Definition 2. (*Fuzzy choice function*). A fuzzy choice function is a function $\mathcal{C}(\sigma): \mathcal{F}(X)^n \to \mathcal{F}(X)$.

Let $\mathcal{C}(\sigma)(x)$ denote the degree to which the group of individuals, N, chooses $x \in X$ given a specific $\sigma \in \mathcal{F}(X)^n$ and fuzzy choice funtion $\mathcal{C}$. It is assumed that $\mathcal{C}(\sigma)$ has *full range*: for any $x \in X$, there exists a $\sigma \in \mathcal{F}(X)^n$ such that $C(\sigma)(x) = \alpha$, for all $\alpha \in [0,1]$.[2] Full range also guarantees that $\mathcal{C}$ is nontrivial, i.e. $\mathcal{C}(\sigma) \neq c$ for all $\sigma \in \mathcal{F}(X)^n$, where c is some constant in $[0,1]$. In addition, $(\sigma_{N\setminus i}, \sigma'_i)$ represents the profile of individual preference functions where $(\sigma_1, \sigma_2, \ldots, \sigma'_i, \ldots, \sigma_n)$ and $(\sigma_{N\setminus i}(x), \sigma_i(x))$ is the profile's restriction to $x \in X$.

[2] Some authors refer to the full range assumption as citizen sovereignty when preferences are required to be single-peaked [3].

The following definitions characterize several properties of fuzzy choice functions.

Definition 3. (*Weakly Paretian*). A fuzzy choice function $\mathcal{C}(\sigma)$ is said to be weakly paretian if for all $\sigma \in \mathcal{F}(X)^n$ and all $x \in X$,

$$\max_{i \in N}(\sigma_i(x)) \geq \mathcal{C}(\sigma)(x) \geq \min_{i \in N}(\sigma_i(x)).$$

In words, weak Paretianism guarrantees that the degree to which a fuzzy choice function selects an alternative is (1) not greater than the choice intensity of the individual who chooses the alternative to the most intense degree and (2) not less than the choice intensity of the individual who chooses the alternative to the least intense degree.

Definition 4. (σ-only). A fuzzy choice function $\mathcal{C}$ is said to satisfy the σ-only condition if for all σ, $\sigma' \in \mathcal{F}(X)^n$ and all $x \in X$ such that $\sigma_i(x) = \sigma_i'(x)$ for all $i \in N$,

$$\mathcal{C}(\sigma)(x) = \mathcal{C}(\sigma')(x).$$

In words, the σ-only condition guarrantees that the degree to which a fuzzy choice function for all $x \in X$ is independent of the choice intensities assigned to other alternatives.

Definition 5. (*Monotonic*). A fuzzy choice function $\mathcal{C}$ is said to be monotonic if, for all $x \in X$, all $\sigma \in \mathcal{F}(X)^n$, and all $\sigma_i' \in \mathcal{F}(X)$,

$$\sigma_i(x) \leq \sigma_i'(x), \forall i \in N \implies \mathcal{C}(\sigma)(x) \leq \mathcal{C}(\sigma_{N \setminus i}, \sigma_i')(x)$$

Monotonicity requires that increasing the degree to which individuals choose a specific alternative will not decrease the degree of social choice for that alternative.

Definition 6. (*Manipulable*). A fuzzy choice function $\mathcal{C}$ is manipulable if there exixts $x \in X$, $\sigma \in \mathcal{F}(X)^n$, $i \in N$ and $\sigma_i' \in \mathcal{F}(X)$ such that

$$
\begin{aligned}
(1)\ \mathcal{C}(\sigma)(x) &< \sigma_i(x) \Rightarrow \mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i},\, \sigma_i')(x)\\
(2)\ \mathcal{C}(\sigma)(x) &> \sigma_i(x) \Rightarrow \mathcal{C}(\sigma)(x) > \mathcal{C}(\sigma_{N\setminus i},\, \sigma_i')(x)
\end{aligned}
$$

According to definition 6, a fuzzy choice function is manipulable if an individual $i \in N$ is able to move the degree of social choice for an alternative in the direction of her sincere choice intesnity of that alternative by submitting a false preference for x.

Definition 7. (*Strategy-Proof (SP)*). A fuzzy choice function $\mathcal{C}$ is said to be strategy-proof if it is not manipulable.

Example. Let $\sigma \in \mathcal{F}(X)^n$ and let $\mathcal{C}$ be a fuzzy choice function. Suppose for some $x \in X$ there exists an $i \in N$ such that $\sigma_i(x) = .4$. Suppose $\mathcal{C}(\sigma)(x) = .3$ and for all $\sigma_i' \in \mathcal{F}(X)$, $\mathcal{C}(\sigma_{N\setminus i}, \sigma_i')(x) = .9$. Hence, $\mathcal{C}$ is manipulable by definition 6 even though $|\sigma_i(x) - \mathcal{C}(\sigma)(x)| < |\sigma_i(x) - \mathcal{C}(\sigma_{N\setminus i}, \sigma_i')(x)|$.

The preceeding example begs the question is a fuzzy choice function manipulable even when it over corrects, of sorts, for manipulation? More specifically, it illustrates a situation where, given Euclidean preferences over the range of the choice function, a fuzzy choice function is manipulable even though the choice function produces an output further away from i's ideal choice instensity when manipulated. Because of this, some authors define SP in the following context [31].

Definition 8. (*Strategy-Proof'*) A fuzzy choice function $\mathcal{C}$ is strategy-proof if for all $x \in X$, all $\sigma \in \mathcal{F}(X)^n$ and all $i \in N$ the following hold for all $\sigma_i' \in \mathcal{F}(X)$:

$$
\begin{aligned}
(1)\ \mathcal{C}(\sigma)(x) &< \sigma_i(x) \Rightarrow \mathcal{C}(\sigma)(x) \geq \mathcal{C}(\sigma_{N\setminus i},\, \sigma_i')(x)\\
(2)\ \mathcal{C}(\sigma)(x) &> \sigma_i(x) \Rightarrow \mathcal{C}(\sigma)(x) \leq \mathcal{C}(\sigma_{N\setminus i},\, \sigma_i')(x)
\end{aligned}
$$

In words, definition 9 guarrantees that when an individual i submits a false preference profile, σ_i, the resulting social choice is "at least as far away" from the original social choice when every individual submits his or her true preference. Hence, individual i has no incentive manipulate the social choice. However, in a formal context, definition 9 and definition 8 are equivalent.

Proposition 9. *Strategy-poofness under definition 7 and strategy-proofness under definition 8 are equivalent.*

Proof. Def. 7 $\implies$ Def. 8. Let $\mathcal{C}$ be a choice function that is strategy-proof under definition 7. Because $\mathcal{C}$ is strategy-proof under definition 7, there does not exist $x \in X$, $\sigma \in \mathcal{F}(X)^n$, $i \in N$ and $\sigma'_i \in \mathcal{F}(X)$ such that $\mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \sigma'_i)(x)$. Thus, for all $x \in X$, $\sigma \in \mathcal{F}(X)^n$, $i \in N$ and $\sigma'_i \in \mathcal{F}(X)$, $\mathcal{C}(\sigma)(x) \geq \mathcal{C}(\sigma_{N\setminus i}, \sigma'_i)(x)$. A symmetrical argument can be made for $\mathcal{C}(\sigma)(x) > \sigma_i(x)$. Because $\sigma \in \mathcal{F}(X)^n$, $x \in X$ and $i \in N$ are arbitrary, strategy-poofness under definition 8 implies strategy-proofness under definition 8.

Def. 8 $\implies$ Def. 7. Let $\mathcal{C}$ be a choice function that is strategy-proof under definition 8. Let $\sigma \in \mathcal{F}(X)^n$, $x \in X$ and $i \in N$ such that $\mathcal{C}(\sigma)(x) < \sigma_i(x)$. Because $\mathcal{C}$ is strategy-proof under definition 9, for all $\sigma'_i \in \mathcal{F}(X)$, $\mathcal{C}(\sigma)(x) \geq \mathcal{C}(\sigma_{N\setminus i}, \sigma'_i)(x)$. Then there does not exist $x \in X$, $\sigma \in \mathcal{F}(X)^n$, $i \in N$ and $\sigma'_i \in \mathcal{F}(X)$ such that $\mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \sigma'_i)(x)$. A symmetrical argument holds for $\mathcal{C}(\sigma)(x) > \sigma_i(x)$. Thus, $\mathcal{C}$ is strategy-proof under definition 7. □

4 Findings

This section details the main findings of the paper. To characterize the properties of fuzzy choice functions, the following formal arguments utilize several reinterpretations of the pivotal voter theorem presented in Reny [32].

Proposition 10. *If a fuzzy choice function $\mathcal{C}$ is strategy-proof, then it satisfies the σ-only condition.*

Proof. Assume $\mathcal{C}$ is SP. Now suppose $\mathcal{C}$ is not σ-only. This proof will show that this leads to a contradiction. By the assumption, there exists σ, $\sigma' \in \mathcal{F}(X)^n$ and an $x \in X$ such that $\sigma(x) = \sigma'(x)$ and $\mathcal{C}(\sigma)(x) \neq \mathcal{C}(\sigma')(x)$.

There are two cases for consideration. Now construct the following profile of individual preferences $Z = \{z_0,\ z_1, ...,\ z_k, ...,\ z_n\}$ such that the following hold:

$$\begin{aligned}
z_0 &= (\sigma_1, ...,\ \sigma_i, ...,\ \sigma_n) \\
z_1 &= (\sigma'_1, ...,\ \sigma_i, ...,\ \sigma_n) \\
&\vdots \\
z_i &= (\sigma'_1, ...,\ \sigma'_i, ...,\ \sigma_n) \\
&\vdots \\
z_n &= (\sigma'_1, ...,\ \sigma'_i, ...,\ \sigma'_n),
\end{aligned}$$

where $z_{i,j}$ signifies $\sigma_j \in z_i$ and $z_{i\setminus i}$ denotes $z_i \setminus \{\sigma_i\}$. It is apparent that there exist some $z_{i-1}, z_i \in Z$ such that $\mathcal{C}(z_{i-1})(x) = \mathcal{C}(\sigma)(x)$ and $\mathcal{C}(z_i)(x) \neq \mathcal{C}(\sigma)(x)$. Without loss of generality, suppose this occurs at $i = 2$. This proof will show that $i = 2$ can manipulate $\mathcal{C}$ at z_1 and z_2. Now there are two cases to consider.

Case 1. $\mathcal{C}(z_1)(x) < \mathcal{C}(z_2)(x)$. First, suppose $\sigma_2(x) < \mathcal{C}(z_2)(x)$. Here, $\mathcal{C}(z_2)(x) > \mathcal{C}(z_{2\setminus 2}, z_{1,2}) = \mathcal{C}(z_1)(x)$. Thus, $i = 2$ can manipulate $\mathcal{C}$ at z_2 by submitting $z_{1,2}$ rather than $z_{2,2}$. Second, suppose $\sigma_2(x) \geq \mathcal{C}(z_2)(x)$. Because $\mathcal{C}(z_1)(x) < \mathcal{C}(z_2)(x)$, then $\sigma_2(x) > \mathcal{C}(z_1)(x)$. However, $\mathcal{C}(z_1)(x) < \mathcal{C}(z_{1\setminus 2}, z_{2,2}) = \mathcal{C}(z_2)(x)$. Thus, $i = 2$ can manipulate $\mathcal{C}$ at z_1 by submitting $z_{2,2}$ rather than $z_{1,2}$. This is a contradiction. Hence, $\mathcal{C}(z_1)(x) \geq \mathcal{C}(z_2)(x)$.

Case 2. $\mathcal{C}(z_1)(x) > \mathcal{C}(z_2)(x)$. First, suppose $\sigma_2(x) < \mathcal{C}(z_1)(x)$. Here, $\mathcal{C}(z_1)(x) > \mathcal{C}(z_{1\setminus 2}.z_{2,2}) = \mathcal{C}(z_2)(x)$. Thus, $i = 2$ can manipulate $\mathcal{C}$ at z_1 by submitting $z_{2,2}$ rather than $z_{1,2}$. Second, suppose $\sigma_2(x) \geq \mathcal{C}(z_1)(x)$. Because $\mathcal{C}(z_1)(x) > \mathcal{C}(z_2)(x)$, then $\sigma_2(x) > \mathcal{C}(z_2)(x)$. However, $\mathcal{C}(z_2)(x) < \mathcal{C}(z_{2\setminus 2}, z_{1,2})(x) = \mathcal{C}(z_2)(x)$, which is another contradiction. Hence,

$\mathcal{C}(z_1)(x) = \mathcal{C}(z_2)(x)$.

The above argument can be replicated for any $i \in N$. Thus, for any $z_i, z_j \in Z$, $\mathcal{C}(z_i)(x) = \mathcal{C}(z_j)(x)$. Accordingly, $\mathcal{C}(\sigma)(x) = \mathcal{C}(z_0)(x) = \mathcal{C}(z_n)(x) = \mathcal{C}(\sigma')(x)$. Hence, $\mathcal{C}$ satisfies the σ-only condition. $\square$

Proposition 11. *If a fuzzy choice function $\mathcal{C}$ is strategy-proof, then it is weakly Paretian.*

Proof. Assume $\mathcal{C}$ is strategy-proof. Now, suppose $\mathcal{C}$ is not weakly Paretian. This proof will show that this leads to a contradiction.There are two cases for consideration.

Case 1. Suppose there exists $x \in X$ and $\sigma \in \mathcal{F}(X)^n$ such that $\mathcal{C}(\sigma)(x) < \min\limits_{i \in N}(\sigma_i(x))$. By full range, we know there also exists a $\sigma' \in \mathcal{F}(X)^n$ such that $\mathcal{C}(\sigma')(x) \geq \min\limits_{i \in N}(\sigma'_i(x))$. Now construct a vector of profiles $Z = (z_0,\ z_1, ...,\ z_i, ...,\ z_n)$ such that

$$\begin{aligned}
z_0 &= (\sigma_1(x), ...,\ \sigma_i(x), ...,\ \sigma_n(x)) \\
z_1 &= (\sigma'_1(x), ...,\ \sigma_i(x), ...,\ \sigma_n(x)) \\
&\vdots \\
z_i &= (\sigma'_1(x), ...,\ \sigma'_i(x), ...,\ \sigma_n(x)) \\
&\vdots \\
z_n &= (\sigma'_1(x), ...,\ \sigma'_i(x), ...,\ \sigma'_n(x)),
\end{aligned}$$

where $z_{i,j}$ signifies $\sigma_j \in z_i$ and $z_{i\backslash i}$ denotes $z_i \backslash \{\sigma_i\}$. It is obvious that there exists $z_{i-1}, z_i \in Z$ such that $\mathcal{C}(z_{i-1})(x) < \mathcal{C}(z_i)(x)$. Suppose $\sigma_i(x) > \mathcal{C}(z_{i-1})(x)$. Then $\mathcal{C}(z_{i-1})(x) < \mathcal{C}(z_{i-1\backslash i}, z_{i,i})(x) = \mathcal{C}(z_i)(x)$. Thus, i can manipulation $\mathcal{C}$ at (z_{i-1}) by submitting $z_{i,i}$ rather than $z_{i-1,i}$. Now suppose $\sigma_i(x) \leq \mathcal{C}(z_{i-1})(x)$, then $\sigma_i(x) < \mathcal{C}(z_i)(x)$. However, $\mathcal{C}(z_i)(x) > \mathcal{C}(z_{i\backslash i}, z_{i-1,i})(x) = \mathcal{C}(z_{i-1})(x)$. Thus, i can manipulate $\mathcal{C}$ at (z_i) by submitting $z_{i-1,i}$ rather than $z_{i,i}$. This is a contradiction.

Case 2. Suppose there exists $x \in X$ and $\sigma \in \mathcal{F}(X)^n$ such that $\mathcal{C}(\sigma)(x) > \max\limits_{i \in N}(\sigma_i(x))$. By full range, we know there also exists a $\sigma' \in \mathcal{F}(X)^n$ such that $\mathcal{C}(\sigma')(x) \leq \max\limits_{i \in N}(\sigma'_i(x))$. Now construct a vector of profiles $Z = (z_0,\ z_1, ...,\ z_i, ...,\ z_n)$ in a manner detailed above. Likewise, there exists $z_{i-1}, z_i \in Z$ such that $\mathcal{C}(z_{i-1})(x) > \mathcal{C}(z_i)(x)$. Suppose $\sigma_i(x) < \mathcal{C}(z_{i-1})(x)$. Then $\mathcal{C}(z_{i-1})(x) > \mathcal{C}(z_{i-1\setminus i}, z_{i,i}) = \mathcal{C}(z_i)(x)$. Thus, i can manipulate $\mathcal{C}$ at (z_{i-1}) by submitting $z_{i,i}$ rather than $z_{i-1,i}$. Now suppose $\sigma_i(x) \geq \mathcal{C}(z_{i-1})(x)$, then $\sigma_i(x) > \mathcal{C}(z_i)(x)$. However, $\mathcal{C}(z_i)(x) < \mathcal{C}(z_{i\setminus i}, z_{i-1,i})(x) = \mathcal{C}(z_{i-1})(x)$, another contradiction.

Hence, $\max\limits_{i\in N}(\sigma_i(x)) \geq \mathcal{C}(\sigma)(x) \geq \min\limits_{i\in N}(\sigma_i(x))$, $\mathcal{C}$ is weakly Paretian. □

Corollary 12. *If a fuzzy choice function $\mathcal{C}$ is strategy-proof but does not satisfy full range then it does not satisfy weak Paretianism.*

Proof. An example will suffice. Suppose, for all $x \in X$ and all $\sigma \in \mathcal{F}(X)^n$, $\mathcal{C}(\sigma)(x) = c$. In this case, $\mathcal{C}$ is strategy-proof but is not weakly Paretian if $c < \sigma_i(x)$ for all $i \in N$. □

Together, proposition 10 and corollary 11 demonstrate the equivalence of weak Paretianism and full range under strategy-proof choice functions. The next proposition and its subsequent corollary highlight the relationship between strategy-proofness and monotonicity.

Proposition 13. *If a fuzzy choice function $\mathcal{C}$ is strategy-proof, then it is monotonic.*

Proof. Assume $\mathcal{C}$ is SP. Now suppose $\mathcal{C}$ is not monotonic. This proof will illustrate that this leads to a contradiction. Because $\mathcal{C}$ is not monotonic, there exists an $x \in X$ and $\sigma,\ \sigma' \in \mathcal{F}(X)^n$ such that $\sigma_i(x) \leq \sigma'_i(x)$, for all $i \in N$, and $\mathcal{C}(\sigma)(x) > \mathcal{C}(\sigma')(x)$. Because $\sigma \neq \sigma'$, there exists at least one $i \in N$ such that $\sigma_i(x) < \sigma'_i(x)$. Now construct the following vector of profiles $Z = (z_0,\ z_1, ...,\ z_i, ...,\ z_n)$ such that

$$
\begin{aligned}
z_0 &= (\sigma_1(x), ..., \sigma_i(x), ..., \sigma_n(x)) \\
z_1 &= (\sigma_1'(x), ..., \sigma_i(x), ..., \sigma_n(x)) \\
&\vdots \\
z_i &= (\sigma_1'(x), ..., \sigma_i'(x), ..., \sigma_n(x)) \\
&\vdots \\
z_n &= (\sigma_1'(x), ..., \sigma_i'(x), ..., \sigma_n'(x)),
\end{aligned}
$$

where $z_{i,j}$ denotes $\sigma_j(x) \in z_i$. Now, there exists a $Z' \subseteq Z$ such that $z_i \in Z'$ if and only if $\sigma_i(x) > \sigma_i'(x)$ for all $i \in N$. For some $z_i \in Z'$, it is obvious that $\mathcal{C}(z_i)(x) < \mathcal{C}(z_{i-1})(x)$, where z_{i-1} is not necessarily in Z'. The proof now shows that i can manipulate $\mathcal{C}$ with two cases. First suppose $\sigma_i(x) > \mathcal{C}(z_i)(x)$. Then, $\mathcal{C}(z_i)(x) < \mathcal{C}(z_{i\setminus i}, z_{i-1,i})$, then i can manipulate $\mathcal{C}$ at z_i by submitting $z_{i-1,i}$ rather than $z_{i,i}$. Second suppose $\sigma_i(x) \leq \mathcal{C}(z_i)(x)$. Then by assumption, $\sigma_i(x) < \mathcal{C}(z_{i-1})(x)$, and $\mathcal{C}(z_{i-1})(x) > \mathcal{C}(z_{i-1\setminus i}, z_{i,i})$. Thus, i can manipulate $\mathcal{C}$ at z_{i-1} by submitting $z_{i,i}$ rather than $z_{i-1,i}$. $\square$

Proposition 11 demonstrates that strategy-proofness is sufficient for a monotonicity. However, previous research in crisp preference relations has shown that strategy-proofness is necessary and sufficient [27]. This is does not hold in the fuzzy framework as the following corollary demonstrates.

Corollary 14. *Montonicity does not imply strategy-proofness.*

Proof. A counter example will suffice. For any $x \in X$ and any $\sigma \in \mathcal{F}(X)^n$, let $\mathcal{C}(\sigma)(x) = \left(\frac{1}{n}\right) \sum_{\forall i \in N} \sigma_i(x)$. It is easy to verify that $\mathcal{C}$ is a monotonic choice function. Now let $x \in X$ and suppose $N = \{1, 2, 3\}$ and $\sigma(x) = (.4, .1, .6)$. In this case, $\mathcal{C}(\sigma)(x) = \frac{1}{3}(.4 + .1 + .6) = .367$. Obviously, some $i \in \{1, 3\}$ could manipulate $\mathcal{C}$ with some $\sigma_i'(x) > \sigma_i(x)$. $\square$

The following defintion is necessary to characterize the domain of strategy-proof choice functions. The proceeding argument applies the same logic as Moulin [26].

Definition 15. (*Augmented median rule*). Let $M : \mathcal{F}(X)^n \to \mathcal{F}(X)$ be a fuzzy choice function defined as follows:

$$M(\sigma)(x) = \mathrm{med}(p_1, ..., \, p_{n-1}, \; \sigma_1(x), ..., \, \sigma_n(x))$$

where $\forall p_i \in [0,1]$.[3]

$\{p_1, ..., \, p_{n-1}\}$ is a set of predefined phantom alternatives that serve two purposes. First, it allows $M(\sigma)$ to be generalized to any type of rank-selecting function such as minimum or maximum. Second, the set also ensures an odd number of alternatives thereby ensuring that the a median can always be selected.

Lemma 16. *$M(\sigma)$ is a strategy-proof fuzzy choice function.*

Proof. Assume $M(\sigma)$ is defined as given above. Suppose $M(\sigma)$ is not strategy-proof. This leads to a contradiction.

There are two cases to cosider. First, let $x \in X$, $i \in N$ and $\sigma \in \mathcal{F}(X)$. Suppose $\sigma_i(x) < M(\sigma)(x)$. Then there exists a $\sigma' \in \mathcal{F}(X)$ such that $M(\sigma)(x) > M(\sigma_{N \setminus i}, \, \sigma_i')$. For clarity, let $M(\sigma)(x) = a$ and $M(\sigma_{N \setminus i}, \, \sigma_i') = b$. Obviously, $a \neq b$ and $a > b$.

Note that $a \in \{p_1, ..., p_{n-1}, \, \sigma_1(x), ..., \, \sigma_i(x), ..., \, \sigma_n(x)\}$. Because $\sigma_i(x) < a$, $\sigma'(x) \not\leq \sigma(x)$, else $\mathrm{med}(p_1, ..., p_{n-1}, \, \sigma_1(x), ..., \, \sigma_i(x), ..., \, \sigma_n(x)) = \mathrm{med}(p_1, ..., p_{n-1}, \, \sigma_1(x), ..., \, \sigma_i'(x), ..., \, \sigma_n(x))$. Thus, $\sigma_i'(x) > \sigma(x)$. This implies then $b = \mathrm{med}(p_1, ..., p_{n-1}, \, \sigma_1(x), ..., \, \sigma_i'(x), ..., \, \sigma_n(x)) \geq a$. However, $M(\sigma_{N \setminus i}, \, \sigma_i') = b$ and $a > b$. This is a contradiction.

Second, suppose $\sigma_i(x) > M(\sigma)(x)$. Then there exists a $\sigma' \in \mathcal{F}(X)$ such that $M(\sigma)(x) < M(\sigma_{N \setminus i}, \, \sigma_i')$. Again, let $M(\sigma)(x) = a$ and $M(\sigma_{N \setminus i}, \, \sigma_i') = b$. Obviously, $a \neq b$ and $a < b$. Because $\sigma_i(x) > a$, $\sigma'(x) \not\geq \sigma(x)$, else $\mathrm{med}(p_1, ..., p_{n-1}, \, \sigma_1(x), ..., \, \sigma_i(x), ..., \, \sigma_n(x)) = \mathrm{med}(p_1, ..., p_{n-1}, \, \sigma_1(x), ..., \, \sigma_i'(x), ..., \, \sigma_n(x))$. Thus, $\sigma_i'(x) < \sigma(x)$. However, as before, $b =$

[3] Several studies characterize the augmented median rule using $\{p_1, ..., \, p_{n+1}\}$ [3, 5, 26]. In this case, however, by setting $p_1 = 0$ and $p_{n+1} = 1$, the rule can be more succinctly written using $n - 1$ alternatives.

$\text{med}(p_1, ..., p_{n-1}, \sigma_1(x), ..., \sigma_i'(x), ..., \sigma_n(x)) \leq a$, a contradiction. The desired result now follows. $\square$

In words, proposition 14 demonstrates that if $i \in N$ attempts to manipulate the value of $M(\sigma)(x)$ with σ_i', one of two events will happen. Either the new manipulated social choice will be identical to the original social choice or the new manipulated social choice will move further away from i's ideal social intensity for $x \in X$. Hence, $i \in N$ will not be better off by reporting any $\sigma_i' \neq \sigma_i$.

The paper's theorem that $\mathcal{C}(\sigma)$ is strategy-proof if and only if $\mathcal{C}(\sigma) = M(\sigma)$ follows the logic in Ching [9] and makes use of the following lemma.

Lemma 17. *Let $\bar{\sigma}_i(x) = 1$ and $\underline{\sigma}_i(x) = 0$ for all $x \in X$ and all $i \in N$. A fuzzy choice function $\mathcal{C}$ is strategy-proof if and only if, for all $\sigma \in \mathcal{F}(X)^n$, all $x \in X$ and all $i \in N$, the following holds:*

$$\mathcal{C}(\sigma)(x) = med\{\sigma_i(x), \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x), \mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x)\}$$

Proof. Suppose $\mathcal{C}$ is a strategy-proof fuzzy choice function. Let $\sigma \in \mathcal{F}(X)^n$ and $x \in X$. By monotonicty and proposition 13, $\mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x) \geq \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$. There are three cases to consider to prove the relationship.

First, suppose $\sigma_i(x) \in (\mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x), \mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x))$. Further, suppose $\mathcal{C}(\sigma)(x) < \sigma_i(x)$, then i can submit $\bar{\sigma}_i(x)$ where $\mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x) > \sigma_i(x) > \mathcal{C}(\sigma)(x)$. Thus, $\mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x)$, and $\mathcal{C}$ is manipulable, a contradiction. Now suppose $\mathcal{C}(\sigma)(x) > \sigma_i(x)$. Similarily, $\mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x) < \sigma_i(x) < \mathcal{C}(\sigma)(x)$. Because $\mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x) < \mathcal{C}(\sigma)(x)$, $\mathcal{C}$ is manipulable, a contradiction. Hence $\mathcal{C}(\sigma)(x) = \sigma_i(x)$ when $\sigma_i(x) \in (\mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x), \mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x))$.

Second, suppose $\sigma_i(x) \leq \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$. To see that $\mathcal{C}(\sigma)(x) = \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$, suppose $\mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$. Then $\underline{\sigma}_i(x) \leq \mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$. In this case, i can manipulate $\mathcal{C}$ at $(\sigma_{N\setminus i}, \underline{\sigma}_i)$ by submitting σ_i rather than $\underline{\sigma}_i$, a contradition. Likewise, suppose $\mathcal{C}(\sigma)(x) > \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x) \geq \sigma_i(x)$. Now, i can manipulate $\mathcal{C}$ at σ by submitting $\underline{\sigma}_i$ rather than σ_i. Hence, $\mathcal{C}(\sigma)(x) = \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$ when $\sigma_i(x) \leq \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$.

Third, suppose $\sigma_i(x) \geq \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$ and $\mathcal{C}(\sigma)(x) > \mathcal{C}(\sigma_{N\setminus i}, \overline{\sigma}_i)(x)$. Then $\bar{\sigma}_i(x) \geq \mathcal{C}(\sigma)(x) > \mathcal{C}(\sigma_{N\setminus i}, \overline{\sigma}_i)(x)$, where i can manipulate $\mathcal{C}$ at $(\sigma_{N\setminus i}, \overline{\sigma}_i)$ by submitting σ_i rather than $\bar{\sigma}_i$. Now suppose $\mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \overline{\sigma}_i)(x)$. Then $\mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \overline{\sigma}_i)(x) \leq \sigma_i(x)$, and i can manipulate $\mathcal{C}$ at σ by submitting $\bar{\sigma}_i$ rather than σ_i. Hence, $\mathcal{C}(\sigma)(x) = \mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x)$ when $\sigma_i(x) \geq \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$.

The preceding arguments prove that if a fuzzy choice function $\mathcal{C}$ is strategy-proof, then $\mathcal{C}(\sigma)(x) = \text{med}\{\sigma_i(x), \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x), \mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x)\}$. Sufficiency is easily obtained from a argument similar to the one in lemma 16. $\square$

Theorem 18. *Any fuzzy collective choice function $\mathcal{C}$ is strategy-proof if and only if it is a fuzzy augmented median voter rule.*

Proof. Once we have established that strategy-proofness implies σ-only (proposition 10) and the relationship in lemma 17, necessity follows from Ching [9]. Sufficieny follows from lemma 16. $\square$

5 Implications for the Spatial Model

Lemma 17 and theorem 18 illustrate that a fuzzy choice function is strategy-proof if and only if it is a form of the fuzzy augmented median rule from definition 15. Further, in contrast to previous results using traditional preference relations, this relationship holds without restricting the domain of individaul preferences, $\mathcal{F}(X)^n$. While the representation of an individual's preferences with the σ_i function clearly produces a transitive preference relations R_i, where $xR_iy \iff \sigma_i(x) \geq \sigma_i(y)$, the use of the σ_i creates substantive differences between the structure of traditional and fuzzy strategy-proof choice functions The reason these differences emerge is the group of individuals are no longer deciding what alternative to select but rather deciding the degree to which the group chooses each alternative.

To illustrate the these difference, the model presented in section 3 can be applied to the spatial model, where the set of alternatives X becomes of subset of k-dimensional Euclidean space or $\mathbb{R}^k$. When $k = 1$, σ_i can be presented by a traditional fuzzy number, i.e. $\sigma_i : \mathbb{R}^1 \to [0, 1]$, which has a similar defintion to a fuzzy subset. Further, it is often assumed that σ_i is *normal*, which requires that there exists $x \in X$ such that $\sigma_i(x) = 1$. In words, normality ensures that every actor views at least one alternative as ideal. While the condition seems innocuous and strongly related to the standard assumptions of spatial models, it is not necessary.

Figure 1 illustrates an three player fuzzy preference profile where each σ_i is represented by a normal fuzzy number in one-dimensional space. It is obvious that the fuzzy number representation allow for greater variation in individual preferences that a traditional single-peaked profile. In this example, not only are the fuzzy preferences able to capute the single-plateau characteristics of concern to some scholars [7, 10, 25], but they also allow for non-single-peaked preferences (player 2), which is one substantive difference between exact and fuzzy choice. Further, the shaded areas show the social choice induced by the fuzzy median rule. To see that the social choice is indeed strategy proof even with non-single-peaked preferences, consider $x_1 \in X$. Here, $\sigma_1(x_1) > 0$, $\sigma_2(x_1) = 0$, and $\sigma_3(x_1) = 0$. Regardless of any $\sigma_1' \in \mathcal{F}(X)$ and any possible values of $\sigma_1'(x_1)$, $M(\sigma)(x_1) = 0$. In addition, consider $x_2 \in X$, where $\sigma_1(x_2) < M(\sigma)(x_2)$, $\sigma_2(x_2) > M(\sigma)(x_2)$, and $\sigma_3(x_2) = M(\sigma)(x_2)$. Similarily, player 1 cannot manipulate the fuzzy choice for x_2. For any $\sigma_1' \in \mathcal{F}(X)$ and any specific value of $\sigma_1'(x_2)$, $M(\sigma_{N\setminus 1}, \sigma_1')(x_2) \geq M(\sigma)(x_2)$. Thus, player 1 can only move the degree of social choice further away from her truthful preference for x_2.

When working in multidimensional space, the framework of the σ_i function remains largely the same, where $\sigma_i : \mathbb{R}^k \to [0, 1]$. When $k = 2$, we are interested in fuzzy subsets where every element in the image of σ_i except $\{0\}$, denoted $\mathrm{Im}(\sigma_i)\setminus\{0\}$, is the interior and boundary of a simple closed curve. A simple closed curve is a curve for which there is a one-to-one continuous function of the unit circle onto it. In addition, a simple closed curve has an interiour that is bounded and an exterior, but there is no need for the curve to be convex. Finally, we can restrict σ_i in particular way such that, for all $t \in \mathrm{Im}(\sigma_i)\setminus\{0\}$, $\{x \in X \mid \sigma_i(x) = t\}$ forms a compact set.

Figure 1: Three-player example of the fuzzy median rule in one-dimensional space

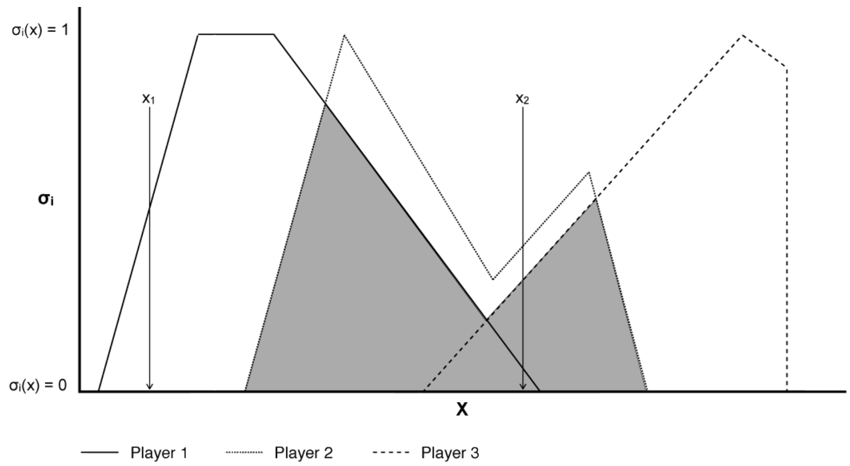
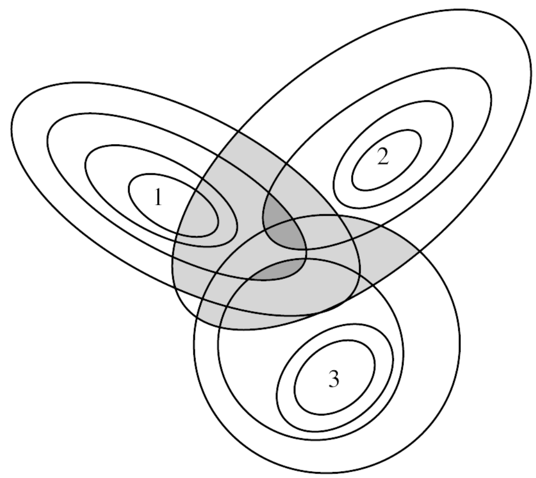


Figure 2 presents a three player fuzzy preference profile in two-dimensional space, and the σ_i function becomes a third dimension perpendicular to both the X and Y policy dimensions. In this case, $\text{Im}(\sigma_i) = \{0, .25, .5, .75, 1.0\}$, where $\sigma_i(x) = 1$ can be represented by individual i's inner-most indifference circe and $\sigma_i(x) = 0$ signifies the area outside i's outer-most indifference circle. When fuzzy preferences are constructed in this manner, they are similar to a Likert scale. As in the previous example, the shaded gray areas show the social choice induced by the fuzzy median rule, and darker areas represent a more intense social choice. Unlike the exact case, the fuzzy median rule remains strategy-proof in two-dimensional without using the dimenson-by-dimenson median rule.

Finally, another substantive difference occurs when no players have intersecting σ_i functions. When this happens, $M(\sigma)(x) = 0$ for all $x \in X$, and the group of playes rejects all possible alternatives. In this case, it is unclear as to what the social choice is. In the traditional appraoch, a choice function associates an alternative to all possible combination of individual preferences. However, in the fuzzy case when the choice function is designating a social choice intensity to each alternative, it is possible that a strategy-proof choice function assigns a zero intensity to all alternatives. This is not necessarily a misrepresentation of the

Figure 2: Fuzzy median rule in two-dimensional space



original intention of strategy-proof choice functions if rejecting all alternatives is some type of social choice.

6 Conclusion

This paper proposed a framework for charaterizing strategy-proof fuzzy choice functions in which individual preferences and the social choice are represented by fuzzy subsets of the set of alternatives. Essentially, actors are deciding to what degree the group chooses each alternative rather than choosing a specific alternative, which is the approach taken in previous studies of both exact and fuzzy social choice. Similar to previous results, strategy-proof fuzzy choice functions satisfy conditions of σ-only, weak Paretianism and monotonicity. In addition, theorem 18 demonstrates a fuzzy choice function is strategy-proof if and only if it is the fuzzy augmented median voter theorem. Unlike previous results, strategy-proof choice functions do not require any restrictions on the consistency of individual preferences or the dimensionality of the alternative space. In fact, section 5 illustrates strategy-proof fuzzy choice when individuals have multi-peaked preferences on a single dimension of alternatives and when the set of alternatives is multidimensional. The results speak to recent debates about the possibility of strategic manipulation of exact choice fucntions with single-peaked preferences. They suggest that when social choice selects alternatives to various degrees there exists strategy-proof choice functions that do not require restrictions on individuals' preferences.

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Mathematical Immortality: Pythagoras' Philosophy of Mathematics and its Persistence through Ancient and Modern Times

Jake Moore

Creighton University

Introduction

"Immortality may be a silly word, but probably a mathematician has the best chance of whatever it may mean" (Hardy 81). G.H. Hardy is one of the great mathematical minds of the 20th century, and he delivered this line in his *A Mathematician's Apology*. What does Hardy mean by this statement? Does he mean that just because someone does mathematics, it gives that person a better chance to be remembered? Not quite. He is talking about mathematical *ideas* being immortal. Unlike other subjects, once a mathematical statement is proven, as long as it is based on propositions and theorems previously shown to be true, there is really no way to disprove it unless the one who proved it made a mathematical error. Mathematics has even corrected some errors in other scientific areas such as astronomy. One clue that our solar system is heliocentric is that the mathematical data obtained through observation did not fit with a geocentric model of the solar system. Hence, Hardy is talking about the theorems of mathematicians that persist throughout all time. Since the ideas and theorems usually bear the names of their discoverers, those names persist as well. The main focus of this paper will be to discuss one of the oldest mathematicians, Pythagoras, and to explain why his philosophy of mathematics has persisted for over 2000 years.

When I mention the name Pythagoras, it would be a safe bet to say that most people think of the Pythagorean Theorem: $a^2 + b^2 = c^2$. This is something that every high school geometry student must (often with great pains) memorize. It is almost as if you are not a true student of geometry until you know the Pythagorean Theorem off the top of your head. Thus, because of an idea that is so central to geometry, Pythagoras' name has lasted some 2500 years. However, this will not be the main focus of this paper. I will discuss why Pythagoras chose to pursue mathematics so fervently. I will answer the following questions: why did Pythagoras think Number was so important? Why did other thinkers in Greece and abroad (Philolaus, Cicero, Plato), both in his time and after his death, consider his ideas worth studying? Lastly, why is it that Pythagoras is considered one of the greatest mathematicians ever, even though many of his ideas seem so simple to us now?

The paper will start with a brief biography of Pythagoras, followed by a discussion on why Pythagoras thought mathematics was worth studying. Following that, I will discuss some of Pythagoras' ideas about mathematics which are of interest to me. Lastly, I will discuss why his ideas can still be important for us today. Before I delve into the meat of the mathematical ideas of Pythagoras, let me first give a brief biography of the man we know as Pythagoras.

A Brief Biography of Pythagoras

Pythagoras was born on the island of Samos around 570 BCE. After many travels through Egypt and Babylonia, he eventually settled in Croton were he founded a community based on philosophical, religious, and political ideas. Much of what we know from Pythagoras comes from the writings of others. Little is known about the views of Pythagoras himself. One thing we do know is that Pythagoras had a reputation for "great learning" (CDC Reeve 15). Two different groups of Pythagoreans emerged within his community: the *akousmatikoi* and the *mathematikoi*. In a nutshell, *akousmatikoi* worshipped the religious teachings of Pythagoras. In addition, they venerated his teachings on the proper way to live. Unlike the *akousmatikoi*, *mathematikoi* had the reputation of studying the philosophical, mathematical, musical, and astronomical aspects of the *kosmos*. (*Mathematikoi* comes from *mathema*, meaning "study" or "learning") To Pythagoras and his followers, all of these different sorts of knowledge were connected, and they believed that "number was the key to understanding the cosmos" (Reeve 15). Since Number is ordered and since the Pythagoreans reasoned that the entire cosmos is ordered in a harmonious arrangement, the Pythagoreans believed that Number was what ordered the universe. Thus, if you understand Number, you can understand the universe. After Pythagoras' death around 494 BCE, there continued to be some Pythagoreans. Philolaus, for instance, is the earliest Pythagorean to leave a work. There also continued to be Pythagoreans in southern Italy, one being Archytas of Tarentum, who was a friend of Plato.

While there is not much known about Pythagoras himself and Pythagoreans seem to have been a rather small sect of philosophers during the 5th through about 2nd centuries BCE, their influence can be seen in other works of mathematics. I will explore this later, but consider the Pythagoreans belief that Number can be used to explain everything in the cosmos. How do physicists today attempt to explain natural and supernatural phenomena? What is seen as the language of the world, or the language that everybody in the world can understand? How do we predict what will happen in the course of events, both on earth and beyond? The answer to all three of these questions is Number (or we might say mathematics). Pythagoras was the first thinker to learn not simply for learning's sake, but to try to understand the world around him. He tried to do so in a rational way, one that looked to *logos* ("reason") and understanding instead of simple observation and primitive teachings. While we still use observation to help us understand our universe, data does not make sense unless it lines up with what is "supposed to happen" or what we "believe should happen."

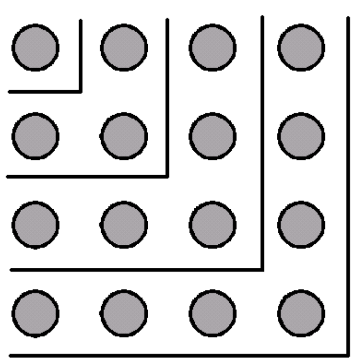
Pythagoras did not have the same approach to mathematics that mathematicians today have. He did not sit down with previously existing theorems, trying to prove new found conjectures or propositions. His thoughts lay more with Number in general. Pythagoras wanted to believe that Number was the key to understanding all things. Number is beautiful and works out many problems for us, so why not think it is the source of our knowledge about the universe?

Pythagoras and Number

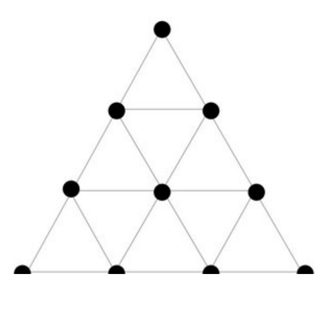
One of the key things to remember about Pythagorean ideas of Number is that, “for the Pythagoreans, Number is a living, qualitative reality which must be approached in an experiential manner,” and it is “not something to be used; rather its nature is to be discovered” (Guthrie 21). David Fideler in his introduction to *The Pythagorean Sourcebook and Library* goes on to consider what modern physics shows us. He points out that the “wavelength” (i.e. a numeric vibrational frequency) tells us how particular electromagnetic energy manifests itself (Guthrie 21). While electromagnetic wavelength does not hit close to home for most of us, we still use mathematical data to give us physical manifestations. Consider this simple example. A person comes up to you and asks how tall your ceiling is. He may want to know because you are buying an entertainment center from him, and he wants to make sure it is not too tall. Without even being in your house, you think about how tall your ceiling is. Since you have never actually measured it before, you use heights that you do know to compare. Well, you think, it is a little bit shorter than a basketball hoop, which is ten feet tall. This tells that man approximately how tall your ceiling is because he can think of it in relation to something that he can imagine. Thus, we have used mathematical data to describe a physical manifestation of something.

An important thing to note now is why Pythagoras believed that Number was so well-ordered and thus why it should be used to described what he first coined as *kosmos* or “world order" (Guthrie 22). The Pythagoreans had a knack for making representations of Number in order to show how ordered it was. Some of his first representations are just words to us, for instance, One, which was Unity, out of which all things arose. Secondly, Two, or Dyad, from which the beginnings of multiplicity, strife, but also *logos* arise (Guthrie). That is, with the onset of Two comes the possibility for disagreement and fighting, since two people, for instance, can have different thoughts about topics. Yet, since there is disagreement or strife, it means that there is the possibility for communication. And with communication one can recite his position on something. Therefore, words must be used to defend a position: reason (*logos*). Along with these representations, there were also a few diagrammatical representations of Number that proved to Pythagoras that Number was the most beautiful thing and thus everything must flow from it.

The first one I will discuss is the square number versus the oblong number. Pythagoras uses these ideas to show that One is Unity and thus the number from which all things flow. A square number results from starting with one dot, and then adding two dots to each subsequent row. An oblong number has the same idea behind it, except you start with two dots. The resulting diagrams give One as a square number because as you keep adding dots, you obtain a square, whereas when you start with two dots and follow the same procedure, you obtain a rectangle. To Pythagoras, a square is a more beautiful shape than an oblong, so something as rationally ordered as the cosmos must have arisen from the idea of a square number.



Another important discovery of Pythagoras' was the Tetraktys. This consists of the first four integers (one, two,



three, four). When you start with one dot, then make two dots below that, three below the two, and finally four below the three you obtain an equilateral triangle with sides of length four units. To Pythagoras, “the Tetraktys symbolized the perfection of Number and the elements which comprise it” (Guthrie 29). Pythagoras takes this a step further with a geometrical approach. He points out that One represents the point, Two represents the line, Three represents the surface, and Four represents the tetrahedron, “the first three-dimensional form” (Guthrie 29). For Pythagoras, this showed how Number is not only linked to itself, but how a manifestation of Number shows the link between zero dimension and the third dimension, or “the manifestation of the first body” (Guthrie 29). Therefore, this Tetraktys (also forming an equilateral triangle) shows how the things that make up the external world can be linked. For Pythagoras, if you are able to use Number to describe the design of the external world so perfectly, then Number must be the root of everything in the cosmos.

Pythagoras’ most basic goal was to gain the most knowledge about the most important things. Here is a key link between Pythagoras’ mathematical ideas and his philosophical ideas. For Pythagoras, “the man who devotes himself to the contemplation of the most beautiful things…may properly be called a philosopher” (Guthrie 30). Of course, Pythagoras is not only talking about mathematics. He believed that the orderliness of the cosmos was the most beautiful of all subjects to study. However, this contemplation also helped a person to better understand himself. Hence, the study of the most beautiful things was a way of life. Pythagoras recognized this, and that is why he was the first person to use the word philosophy (or Greek *philosophia*, meaning “wisdom”).

These were just a few reasons why Pythagoras thought mathematics was the most beautiful subject to study. As the most beautiful, it also fit the bill to be the one thing that could describe the cosmos perfectly. However, this will not be the focus of the next part of the paper. In his biography of Pythagoras, Iamblichus mentions Pythagoras' thoughts on studying in general. Even though Iamblichus does not often mention mathematics, it is clear that when he mentions Pythagoras' beliefs on studying, mathematics is at the heart of the discussion. So, we turn to Iamblichus.

Iamblichus and Pythagoras

Pythagoras studied many disciplines besides mathematics. Whether Pythagoras was studying math, astronomy, or politics, Iamblichus says that Pythagoras believed "that disciplines and studies should imply some kind of labor" (Guthrie 73). Most forms of study imply labor. Biologists observe organisms for hours every day in order to understand their processes of survival, political scientists study years of data in order to discern voting patterns, and pianists spend hours at a piano learning notes and movements in order to perform perfectly at recitals. This is why Pythagoras studied subjects other than math: they were challenging. Still, mathematics implies a kind of labor different from other subjects. In mathematics there are not too many discoveries that happen all of a sudden. When someone finally proves a proposition and it becomes a theorem, she is not usually the person who came up with the proposition. In fact, the proposition was usually put forth many years earlier, but the person who proposed it did not have the mathematical tools at his disposal to prove it. The labor in mathematics involves not just one person coming up with a fresh, interesting proposition but also involves many others years later developing new mathematical techniques that can aid in proving the proposition. This is part of the labor involved in mathematics that leads me to a better understanding of why Pythagoras devoted so much time to studying Number.

Another thing involved in the labor of mathematics is trying to make the ideas accessible to others. A mathematician can come up with a new theorem, but often only a handful of people in the entire world have any idea what she is talking about. Thus, the mathematician must come up with analogies and symbols in order to explain her discovery. According to Iamblichus, Pythagoras was one of the first people to use symbols in his teachings. Pythagoras says that when difficult meanings are "made clear and obvious even to the crowds, then they will be found analogous to prophetic sayings" (Guthrie 83). Symbology in mathematics is essential to its being a universal language. Even though Pythagoras never really said anything of this sort, I think he would agree with people who talk about math as a universal language. Consider a different subject: biology. If an American biologist wants to discuss functions of the liver with a French biologist, he cannot use the word "liver." He can point to a picture of the liver when he talks, but only then will the French biologist then associate that picture with the French word for "liver," (*foie*) and understand the conversation. In contrast with mathematical symbols, every mathematician in the world understands by the symbol of an upside-down "A" in a proof, every element of the set is in question. Whether she is an American, Chinese, or German mathematician, she will know that an upside-down "A" means this.

Using symbols and analogies makes mathematics more understandable to those who do not know the first thing about math. Although not stated explicitly, I would say Iamblichus makes it pretty clear that Pythagoras thought one of the key things in studying was to find a way to make your studies accessible to the masses. Iamblichus makes clear one reason why Pythagoras was treated like a god in his time. Iamblichus tells of Pythagoras' ability to predict an earthquake from the taste of the water tasted in a well. While Pythagoras probably did not do this, this account of Iamblichus can be seen as an early account of the predictive power of mathematics. Things like earthquakes, hurricanes, and even meteorite crashes can be predicted by mathematics, along with observation. This, I would say, is another part of the laborious nature of Number. In order to predict phenomena accurately, mathematicians must have years of study at their disposal. In addition, mathematicians must practice using their theorems to make predictions, and they tweak them if they fail by even the slightest bit. Mathematicians desire to be so accurate (and necessarily must be extremely accurate most of the time) that thousands upon thousands of mathematical tests are needed before a theorem is accepted and put into practice. Yet, this laborious nature of Number is what I think Pythagoras would say is one of the things that makes mathematics beautiful. Also, he would say that the fact that the laborious nature of Number leads to results often easily explicable, or things that look much simpler than those you started out with, shows the beauty of the subject, and makes it unique among other subjects.

A Few More Sentences from Iamblichus

In addition to his biography of Pythagoras, Iamblichus also reported some sentences from Pythagoras in his *Exhortation to Philosophy*. These sentences show that Pythagoras wished to argue using reason (*logos*) instead of flowery sentences (rhetoric). This is what mathematics is all about. If I wish to present a new mathematical proof, I do not use flowery language or ornate sentences to get my point across. Rather, I use short, to-the-point language that will make my proof the shortest and clearest that it possibly can be. In fact, one of Iamblichus' sentences states this explicitly. He writes, "Pythagoras said, 'say not few things in many words, but much in few words'" (Guthrie 273). When mathematicians sit down to write their proofs, this is exactly what they are thinking. We know Pythagoras studied mathematics because he believed it to be the most beautiful of subject. Math would not be beautiful if mathematicians provided long-winded speeches that proved very little or contributed nothing to mathematics as an academic discipline.

Although it seems Pythagoras meant this statement to describe how one should approach life in general, I think it is clear that it is one of the main reasons he studied mathematics.

Another sentence deals with making and correcting mistakes. Iamblichus writes, “Pythagoras said: endeavor not to conceal your errors by words, but to remedy them by reproofs” (Guthrie 273). Using words to cover up a mathematical mistake does not work. In mathematical work, all someone has to do is show one counterexample to a theorem to refute it. Thus, the only way to remedy your mistake is to go back and fix it (i.e. write a new proof). Pythagoras believed that this contributed to the learning process. One does not learn if, when he makes mistakes, he tries to think of language to cover them up. He learns more when he goes back to his work to see what he did wrong. Pythagoras would say that a lover of wisdom (philosopher desires to know the truth. The truth about Number does not manifest itself in flowery language that makes up for a mistake. Truth comes from the search for knowledge which, in turn, often requires working on a problem over and over again, checking it numerous times, and each time going back to the problem to find the mistakes you made so that you will not make them again. This is true of any subject, but the most obvious subject to think about this in action is math.

Conclusion

As I started with a quotation from G.H. Hardy, it seems only fitting to end with one as well. In his *Apology*, Hardy asserts that

> The ‘seriousness’ of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the *significance* of the mathematical ideas which it connects. We may say, roughly, that a mathematical idea is ‘significant’ if it can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas (Hardy 89).

If this is the litmus test for deciding whether Pythagoras’ mathematical ideas are significant, I would say that they pass with flying colors. When I speak of his ideas, I do not refer to formulas like the Pythagorean Theorem. While this is a nice result, its only purpose is it to find the sides of a right triangle, and how often are mathematicians really looking to find the sides of a right triangle? I would argue that the reason Pythagoras' ideas pass Hardy's test is because of the way Pythagoras approached mathematics. He wanted to study mathematics in a way that both made it more accessible to the general public and obtained much knowledge for the learner of mathematics. More specifically, Pythagoras believed that Number was the key to understanding the universe. If something so rationally ordered and beautiful as mathematics could be used to describe our cosmos, then this would strongly imply that the cosmos is the most beautiful, ordered, rational place that human beings could possibly dwell in. Today, when we want to describe how the world works, investigators often turn to mathematics. This is a movement that started with Pythagoras and I think it is fairly clear that ideas which have persisted for 2500 years can rightly be called significant in Hardy's terms.

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A FUZZY MATHMATICAL MODEL FOR UNITED STATES ECONOMIC GROWTH

Christopher S. Carlson
Department of Chemistry, Creighton University
ChristopherCarlson@creighton.edu

1. Introduction

Economic growth in the United States has fluctuated greatly since the 17th century when the colonies were first established. Over this time the United States has risen to become the largest economy in the world. Despite the continual growth in the economy government policy, private sector factors, and communal influences have shaped the economy. The overall growth of the economy will be judged on the change in the Real Gross Domestic Product (GDP). The primary task is to develop an objective strategy to deal with the factors that effect U.S. economic growth by analyzing the economy broken up into four-year terms based on presidential term periods since 1960. This year was chosen as the base year because of the amount of readily available information beginning at this time. 1960 marks the end of the "Post-War era" and is also marked by John F. Kennedy's historic tax cuts, which brought tax levels to a rate comparable to today's rates. This allows for more accurate comparison.

The overarching goal of this project is to determine which conditions promote US economic growth measured by the percentage change in real Gross Domestic Product (GDP). This overarching goal was then broken down into three components designed to help promote the goal of U.S. economic growth.

G1: Government Policy- It should be the intent of the government to pass laws which create favorable conditions for economic growth.

G2: Private Sector- It should be the intent of the private sector to earn a profit and expand thus increasing GDP while also increasing the number of jobs created. The private sector employs the majority of the employees in the United States making in one of the most influential factors in economic growth.

G3: Community- The economy will grow more rapidly when the population is healthy and happy making individuals more willing to contribute to the work force.

These components were then broken down into further sub categories. This allowed the components of the economic growth to be measured based on multiple factors. Government policy was broken down into six subcategories:

G_{11}: Taxation- As tax rates are increased less money goes into the economy however more money is directed toward government expenditures which also contribute to the overall health of the economy. This sub factor will be measure by the income tax of the highest tax bracket.

G_{12}: Monetary Policy- The growth of the economy is affected by money supply and interest rates by the central bank. Although this is not indirect control of the government and is instead controlled by the Federal Reserve, for ease of comparison this will be used as a sub factor to government policy. Inflation rate will be used as measurement of this.

G_{13}: Government Spending- Government spending contributes to the overall size and growth of the economy by increasing or decreasing the amount of money the government contributes to the economy. This will be measured by the increase in the national debt level by year.

G_{14}: Environmental Regulation- As environmental laws are passed they protect the environment and maintain the sanctity of our natural resources. However these regulations also affect the growth of the economy by costing business money and time decreasing their ability to contribute to economic growth. Expert opinion will be used to measure this on a year to year basis.

G_{15}: Trade Regulations- As these laws are passed they often are passed with the intent of increasing exports and decreasing imports in theory, helping the United States economy. The number of these trade regulations will be measured by expert opinion on a year to year basis.

G_{16}: International Conflict- Wars effect economic growth by drawing away resources from American investment as well as manpower and public morale. The amount of international conflict will be measured using military spending as a percent of GDP.

The private sector was also broken down into four categories:

G_{21}: Trade- Trade is one of the largest factors contributing to economic growth. The more the United States can export the more money that is brought into the country. Importing allows for specialization which increases the efficiency and productivity of the economy. The measurement of trade will be the amount of exports as a percent of GDP.

G_{22}: Technological Advancement- As new technologies arise they increase efficiency in the economy and allow for faster production. These changes allow for the economy to grow. This will be measured by expert opinion by year and the number of new technologies created.

G_{23}: Expectations- expectations toward the future about tax increases or decreases, of job security, or of new technological advancements all effect a consumers willingness to spend. These all effect the growth rate of the economy. This will be measured by expert opinion on a year to year basis on the populations perspective of future economic conditions.

G_{24}: Private Investment- Private investment contributes to the invention of new technologies, and the growth of businesses. As private investment goes up normally so does the growth in businesses. This will be measured as private investment as a percent of GDP or by expert opinion if sufficient data cannot be acquired.

Community will be broken down into six categories:

G_{31}: Crime Rate- As the crime rate falls fewer resources have to be spent on protection and repairs. The number of murders as a percent of population will measure this.

G_{32}: Health- the overall health of the population means that more people can contribute to economic growth. This will be measured by life expectancy at birth.

G_{33}: Family Life- The stability of family life will indicate both population morale and the stability. This will be measured by the divorce rate.

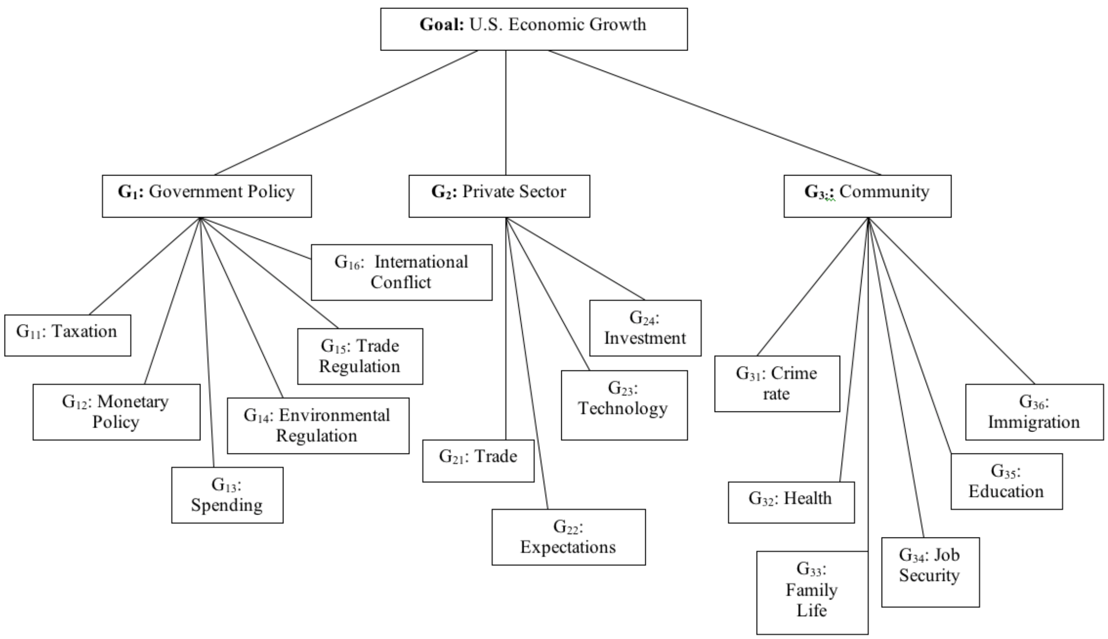
G_{34}: Job Security- As the economy grows jobs become more stable and the unemployment rate will decrease.

G_{35}: Education- As more people become higher educated they develop more technologies and increase specialization. This will be measured by percent of college graduates as a percent of population.

G_{36}: Immigration- As more people immigrate to the United States the size of the work force grows allowing for greater production and economic growth. This will be measured by the number of first generation immigrants as a percent of population.

1.2 Model

A diagram of the goals and sub factors is shown below:



2.1 Expert Opinion

Six experts, E_j were asked to rank the importance of each goal, G_i, as to their importance with respect to the their importance of achieving the overarching goal of U.S. economic growth. The experts pooled are as follows:

E_1: Professor Tim Bastian,
Creighton University, Instructor of Economics
bastian@creighton.edu

E_2: Dr. John Deskins,
Creighton University, Assistant Professor of Economics
JohnDeskins@creighton.edu

E_3: Dr. Vasudeva Murthy
Creighton University, Professor of Economics

vmurthy@creighton.edu

E_4: Professor Ernie Goss
Creighton University, Professor of Economics
ernieg@creighton.edu

E_5: Professor Art Diamond
University of Nebraska at Omaha, Professor of Economics
adiamond@unomaha.edu

E_6: Professor Mark Wohar
University of Nebraska at Omaha, Professor of Economics
mwohar@mail.unomaha.edu

2.2 Expert Rankings

The experts ranked their opinions as follows:

	E_1	E_2	E_3	E_4	E_5	E_6
G_1	9	9	9	10	1	9
G_2	10	10	9	10	2	8
G_3	5	9	8	10	3	6

The experts were then poled on the importance of the sub factors G_{ij}, in attaining G_i. Their rankings are as follows:

	E_1	E_2	E_3	E_4	E_5	E_6
G_{11}	9	7	10	8	10	8
G_{12}	10	8	9	7	8	10
G_{13}	7	NR*	4	7	1	9
G_{14}	8	7	8	7	5	4
G_{15}	9	5	4	10	1	7
G_{16}	6	9	8	10	5	7

*E_2 chose not to give a response to the importance of G_{13}.

	E_1	E_2	E_3	E_4	E_5	E_6
G_{21}	9	8	9	8	5	6
G_{22}	7	10	10	7	10	8
G_{23}	10	8	8	6	4	7
G_{24}	10	10	9	8	9	7

	E_1	E_2	E_3	E_4	E_5	E_6

G_{31}	4	10	9	8	10	3
G_{32}	3	10	9	6	5	6
G_{33}	3	9	10	5	4	7
G_{34}	6	7	8	8	1	7
G_{35}	7	9	9	8	2	10
G_{36}	6	7	8	5	3	6

3.1 The Linear Equations

The experts were weighted in order to obtain the linear equations:

	E_1	**E_2**	**E_3**	**E_4**	**E_5**	**E_6**	**Row Average**	**Goal Weight**
G_1	9	9	9	10	1	9	7.833	0.343
G_2	10	10	9	10	2	8	8.167	0.358
G_3	5	9	8	10	3	6	6.833	0.299
Total	24	28	26	30	6	23	22.833	

These goal weights gave the following equation:

$$P= (.343)G_1+(.358)G_2+(.299)G_3$$

This was repeated for each of the G_{ij}'s of the G_i's:

	E_1	**E_2**	**E_3**	**E_4**	**E_5**	**E_6**	**Row Average**	**Goal Weight**
G_{11}	9	7	10	8	10	8	8.667	0.202
G_{12}	10	8	9	7	8	10	8.667	0.202
G_{13}	7		4	7	1	9	5.600	0.130
G_{14}	8	7	8	7	5	4	6.500	0.151
G_{15}	9	5	4	10	1	7	6.000	0.140
G_{16}	6	9	8	10	5	7	7.500	0.175
Total	49	36	43	49	30	45	42.933	

Giving the formula:

$$P_1= (.202)G_{11} + (.202)G_{12} + (.130)G_{13} + (.151)G_{14} + (.140)G_{15} + (.175)G_{16}$$

	E_1	**E_2**	**E_3**	**E_4**	**E_5**	**E_6**	**Row Average**	**Goal Weight**
G_{21}	9	8	9	8	5	6	7.500	0.233
G_{22}	7	10	10	7	10	8	8.667	0.269
G_{23}	10	8	8	6	4	7	7.167	0.223
G_{24}	10	10	9	8	9	7	8.833	0.275
Total	36	36	36	29	28	28	32.167	

Giving the formula:

$$P_2 = (.233)G_{21} + (.269)G_{22} + (.223)G_{23} + (.275)G_{24}$$

	E_1	E_2	E_3	E_4	E_5	E_6	Row Average	Goal Weight
G_{31}	4	10	9	8	10	3	7.333	0.185
G_{32}	3	10	9	6	5	6	6.500	0.164
G_{33}	3	9	10	5	4	7	6.333	0.160
G_{34}	6	7	8	8	1	7	6.167	0.155
G_{35}	7	9	9	8	2	10	7.500	0.189
G_{36}	6	7	8	5	3	6	5.833	0.147
Total	29	52	53	40	25	39	39.667	

Giving the formula:

$$P_3 = (.185)G_{31} + (.164)G_{32} + (.160)G_{33} + (.155)G_{34} + (.189)G_{35} + (.147)G_{36}$$

These equations can then combined by substituting P_i in for G_i in P. Giving the following equation:

$$P = (.343)P_1 + (.358)P_2 + (.299)P_3$$

$$P = (.343)[(.202)G_{11} + (.202)G_{12} + (.130)G_{13} + (.151)G_{14} + (.140)G_{15} + (.175)G_{16}] + (.358)[(.233)G_{21} + (.269)G_{22} + (.223)G_{23} + (.275)G_{24}] + (.299)[(.185)G_{31} + (.164)G_{32} + (.160)G_{33} + (.155)G_{34} + (.189)G_{35} + (.147)G_{36}]$$

3.2 The Weighted Linear Equations

The row averages from the previous matrices were then used to yield the weights of G_j giving the following matrices:

	E_1	E_2	E_3	E_4	E_5	E_6	Row Average
G_1	0.375	0.321	0.346	0.333	0.167	0.391	0.322
G_2	0.417	0.357	0.346	0.333	0.333	0.348	0.356
G_3	0.208	0.321	0.308	0.333	0.500	0.261	0.322

These weighted row averages can then be combined to give the following equation, P_w, where w=weighted.

$$P_w = (.322)G_1 + (.356)G_2 + (.322)G_3$$

This process was then repeated for each one group of sub factors, G_{ij}'s for each G_i yielding the following:

	E_1	E_2	E_3	E_4	E_5	E_6	Row Average
G_{11}	0.184	0.194	0.233	0.163	0.333	0.178	0.214
G_{12}	0.204	0.222	0.209	0.143	0.267	0.222	0.211
G_{13}	0.143	NA	0.093	0.143	0.033	0.200	0.122
G_{14}	0.163	0.194	0.186	0.143	0.167	0.089	0.157
G_{15}	0.184	0.139	0.093	0.204	0.033	0.156	0.135
G_{16}	0.122	0.250	0.186	0.204	0.167	0.156	0.181

$P_{w1}= (.214)G_{11} + (.211)G_{12} + (.122)G_{13} + (.157)G_{14} + (.135)G_{15}+ (.181)G_{16}$

	E_1	E_2	E_3	E_4	E_5	E_6	Row Average
G_{21}	0.250	0.222	0.250	0.276	0.179	0.214	0.232
G_{22}	0.194	0.278	0.278	0.241	0.357	0.286	0.272
G_{23}	0.278	0.222	0.222	0.207	0.143	0.250	0.220
G_{24}	0.278	0.278	0.250	0.276	0.321	0.250	0.275

$P_{w2}= (.232)G_{21} + (.272)G_{22} + (.220)G_{23} + (.275)G_{24}$

	E_1	E_2	E_3	E_4	E_5	E_6	Row Average
G_{31}	0.138	0.192	0.170	0.200	0.400	0.077	0.196
G_{32}	0.103	0.192	0.170	0.150	0.200	0.154	0.162
G_{33}	0.103	0.173	0.189	0.125	0.160	0.179	0.155
G_{34}	0.207	0.135	0.151	0.200	0.040	0.179	0.152
G_{35}	0.241	0.173	0.170	0.200	0.080	0.256	0.187
G_{36}	0.207	0.135	0.151	0.125	0.120	0.154	0.149

$P_{w3}= (.196)G_{31} + (.162)G_{32} + (.155)G_{33} + (.152)G_{34} + (.187)G_{35} + (.149)G_{36}$

The P_{wi}'s can then be combined into P_w to replace the G_i's to give the formula:

$P_w= (.322)G_1 + (.356)G_2 + (.322)G_3$

$P_w= (.322)P_1 + (.356)P_2 + (.322)P_3$

$P= (.322)[(.214)G_{11} + (.211)G_{12} + (.122)G_{13} + (.157)G_{14} + (.135)G_{15}+ (.181)G_{16}] + (.356)[(.232)G_{21} + (.272)G_{22} + (.220)G_{23} + (.275)G_{24}] + (.322)[(.196)G_{31} + (.162)G_{32} + (.155)G_{33} + (.152)G_{34} + (.187)G_{35} + (.149)G_{36}]$

4. Fuzzy Preference Relations

Using fuzzy binary relations on the five experts preferences of each of the G_i's. The lower number represent the experts greater preference toward that G_i over the other G_i's. This was done using:

$$p_k(Gi,Gj)=\begin{cases}(Gi-Gj+.5)\wedge 1 & if\ Gi\geq Gj,\\ 1-[Gi-Gj+.5\wedge 1] & if\ Gi<Gj,\end{cases}$$

Resulting in the following R_i matrices, where I denotes the expert:

R_1		**G_1**	**G_2**	**G_3**
	G_1	0.500	0.542	0.333
	G_2	0.458	0.500	0.292
	G_3	0.667	0.708	0.500

R_2		**G_1**	**G_2**	**G_3**
	G_1	0.500	0.536	0.500
	G_2	0.464	0.500	0.464
	G_3	0.500	0.536	0.500

R_3		**G_1**	**G_2**	**G_3**
	G_1	0.500	0.500	0.462
	G_2	0.500	0.500	0.462
	G_3	0.538	0.538	0.500

R_4		**G1**	**G2**	**G3**
	G1	0.500	0.500	0.500
	G2	0.500	0.500	0.500
	G3	0.500	0.500	0.500

R_5		G1	G2	G3
	G1	0.500	0.667	0.833
	G2	0.333	0.500	0.667
	G3	0.167	0.333	0.500

R_6		**G_1**	**G_2**	**G_3**
	G_1	0.500	0.457	0.370
	G_2	0.543	0.500	0.413
	G_3	0.630	0.587	0.500

This process was repeated for each of the sub factors for each expert using the same guidelines. R_i^{Gj}, where i is the expert and j is the goal being analyzed. Yielding the following matrices:

Matrices for G_1:

R_1^{G1}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}
G_{11}	0.500	0.520	0.459	0.480	0.500	0.439
G_{12}	0.480	0.500	0.439	0.459	0.480	0.418
G_{13}	0.541	0.561	0.500	0.520	0.541	0.480
G_{14}	0.520	0.541	0.480	0.500	0.520	0.459
G_{15}	0.500	0.520	0.459	0.480	0.500	0.439
G_{16}	0.561	0.582	0.520	0.541	0.561	0.500

R_2^{G1}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}
G_{11}	0.500	0.528	NA	0.500	0.444	0.556
G_{12}	0.472	0.500	NA	0.472	0.417	0.528
G_{13}	NA	NA	NA	NA	NA	NA
G_{14}	0.500	0.528	NA	0.500	0.444	0.556
G_{15}	0.556	0.583	NA	0.556	0.500	0.611
G_{16}	0.444	0.472	NA	0.444	0.389	0.500

R_3^{G1}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}
G_{11}	0.500	0.477	0.360	0.453	0.360	0.453
G_{12}	0.523	0.500	0.384	0.477	0.384	0.477
G_{13}	0.640	0.616	0.500	0.593	0.500	0.593
G_{14}	0.547	0.523	0.407	0.500	0.407	0.500
G_{15}	0.640	0.616	0.500	0.593	0.500	0.593
G_{16}	0.547	0.523	0.407	0.500	0.407	0.500

R_4^{G1}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}
G_{11}	0.500	0.480	0.480	0.480	0.541	0.541
G_{12}	0.520	0.500	0.500	0.500	0.561	0.561
G_{13}	0.520	0.500	0.500	0.500	0.561	0.561
G_{14}	0.520	0.500	0.500	0.500	0.561	0.561
G_{15}	0.459	0.439	0.439	0.439	0.500	0.500
G_{16}	0.459	0.439	0.439	0.439	0.500	0.500

R_5^{G1}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}
G_{11}	0.500	0.433	0.200	0.333	0.200	0.333
G_{12}	0.567	0.500	0.267	0.400	0.267	0.400
G_{13}	0.800	0.733	0.500	0.633	0.500	0.633
G_{14}	0.667	0.600	0.367	0.500	0.367	0.500
G_{15}	0.800	0.733	0.500	0.633	0.500	0.633
G_{16}	0.667	0.600	0.367	0.500	0.367	0.500

R_6^{G1}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}
G_{11}	0.500	0.544	0.522	0.411	0.478	0.478
G_{12}	0.456	0.500	0.478	0.367	0.433	0.433
G_{13}	0.478	0.522	0.500	0.389	0.456	0.456

G_{14}	0.589	0.633	0.611	0.500	0.567	0.567
G_{15}	0.522	0.567	0.544	0.433	0.500	0.500
G_{16}	0.522	0.567	0.544	0.433	0.500	0.500

Matrices for G_2:

R_1^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0.500	0.444	0.528	0.528
G_{22}	0.556	0.500	0.583	0.583
G_{23}	0.472	0.417	0.500	0.500
G_{24}	0.472	0.417	0.500	0.500

R_2^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0.500	0.556	0.500	0.556
G_{22}	0.444	0.500	0.444	0.500
G_{23}	0.500	0.556	0.500	0.556
G_{24}	0.444	0.500	0.444	0.500

R_3^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0.500	0.528	0.472	0.500
G_{22}	0.472	0.500	0.444	0.472
G_{23}	0.528	0.556	0.500	0.528
G_{24}	0.500	0.528	0.472	0.500

R_4^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0.500	0.466	0.431	0.500
G_{22}	0.534	0.500	0.466	0.534
G_{23}	0.569	0.534	0.500	0.569
G_{24}	0.500	0.466	0.431	0.500

R_5^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0.500	0.679	0.464	0.643
G_{22}	0.321	0.500	0.286	0.464
G_{23}	0.536	0.714	0.500	0.679
G_{24}	0.357	0.536	0.321	0.500

R_6^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0.500	0.571	0.536	0.536
G_{22}	0.429	0.500	0.464	0.464
G_{23}	0.464	0.536	0.500	0.500
G_{24}	0.464	0.536	0.500	0.500

Matrices for G_3:

R_1^{G3}	G_{31}	G_{32}	G_{33}	G_{34}	G_{35}	G_{36}
G_{31}	0.500	0.466	0.466	0.569	0.603	0.569
G_{32}	0.534	0.500	0.500	0.603	0.638	0.603

G_{33}	0.534	0.500	0.500	0.603	0.638	0.603
G_{34}	0.431	0.397	0.397	0.500	0.534	0.500
G_{35}	0.397	0.362	0.362	0.466	0.500	0.466
G_{36}	0.431	0.397	0.397	0.500	0.534	0.500

R_2^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0.500	0.500	0.481	0.442	0.481	0.442
G_{32}	0.500	0.500	0.481	0.442	0.481	0.442
G_{33}	0.519	0.519	0.500	0.462	0.500	0.462
G_{34}	0.558	0.558	0.538	0.500	0.538	0.500
G_{35}	0.519	0.519	0.500	0.462	0.500	0.462
G_{36}	0.558	0.558	0.538	0.500	0.538	0.500

R_3^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0.500	0.500	0.481	0.442	0.481	0.442
G_{32}	0.500	0.500	0.481	0.442	0.481	0.442
G_{33}	0.519	0.519	0.500	0.462	0.500	0.462
G_{34}	0.558	0.558	0.538	0.500	0.538	0.500
G_{35}	0.519	0.519	0.500	0.462	0.500	0.462
G_{36}	0.558	0.558	0.538	0.500	0.538	0.500

R_4^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0.500	0.450	0.425	0.500	0.500	0.425
G_{32}	0.550	0.500	0.475	0.550	0.550	0.475
G_{33}	0.575	0.525	0.500	0.575	0.575	0.500
G_{34}	0.500	0.450	0.425	0.500	0.500	0.425
G_{35}	0.500	0.450	0.425	0.500	0.500	0.425
G_{36}	0.575	0.525	0.500	0.575	0.575	0.500

R_5^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0.500	0.300	0.260	0.140	0.180	0.220
G_{32}	0.700	0.500	0.460	0.340	0.380	0.420
G_{33}	0.740	0.540	0.500	0.380	0.420	0.460
G_{34}	0.860	0.660	0.620	0.500	0.540	0.580
G_{35}	0.820	0.620	0.580	0.460	0.500	0.540
G_{36}	0.780	0.580	0.540	0.420	0.460	0.500

R_6^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0.500	0.577	0.603	0.603	0.679	0.577
G_{32}	0.423	0.500	0.526	0.526	0.603	0.500
G_{33}	0.397	0.474	0.500	0.500	0.577	0.474
G_{34}	0.397	0.474	0.500	0.500	0.577	0.474
G_{35}	0.321	0.397	0.423	0.423	0.500	0.397
G_{36}	0.423	0.500	0.526	0.526	0.603	0.500

4.2 Cores

The core is the set of dominated goals, those that are defeated by the pairwise comparisons by a majority. The following will determine the core:

$$h_{ij}^{k} = \begin{cases} 1\, if\ \ r_{ij}^{k} < .5, \\ 0\ otherwise \end{cases}$$

Giving the following matrices where R_i denotes the expert:

R_1	**G_1**	**G_2**	**G_3**
G_1	0	0	1
G_2	1	0	1
G_3	0	0	0

R_2	**G_1**	**G_2**	**G_3**
G_1	0	0	0
G_2	1	0	1
G_3	0	0	0

R_3	**G_1**	**G_2**	**G_3**
G_1	0	0	1
G_2	0	0	1
G_3	0	0	0

R_4	**G_1**	**G_2**	**G_3**
G_1	0	0	0
G_2	0	0	0
G_3	0	0	0

R_5	**G_1**	**G_2**	**G_3**
G_1	0	0	0
G_2	1	0	0
G_3	1	1	0

R_6	**G_1**	**G_2**	**G_3**
G_1	0	1	1
G_2	0	0	1
G_3	0	0	0

This process was the repeated for the sub factors, G_{ij}, for each of the G_i's. Matrices for G_1:

R_1^{G1}	**G_{11}**	**G_{12}**	**G_{13}**	**G_{14}**	**G_{15}**	**G_{16}**
G_{11}	0	0	1	1	0	1
G_{12}	1	0	1	1	1	1

G_{13}	0	0	0	0	0	1
G_{14}	0	0	1	0	0	1
G_{15}	0	0	1	1	0	1
G_{16}	0	0	0	0	0	0

R_2^{G1}	**G_{11}**	**G_{12}**	**G_{13}**	**G_{14}**	**G_{15}**	**G_{16}**
G_{11}	0	0	0	0	1	0
G_{12}	1	0	0	1	1	0
G_{13}	0	0	0	0	0	0
G_{14}	0	0	0	0	1	0
G_{15}	0	0	0	0	0	0
G_{16}	1	1	0	1	1	0

R_3^{G1}	**G_{11}**	**G_{12}**	**G_{13}**	**G_{14}**	**G_{15}**	**G_{16}**
G_{11}	0	1	1	1	1	1
G_{12}	0	0	1	1	1	1
G_{13}	0	0	0	0	0	0
G_{14}	0	0	1	0	1	0
G_{15}	0	0	0	0	0	0
G_{16}	0	0	1	0	1	0

R_4^{G1}	**G_{11}**	**G_{12}**	**G_{13}**	**G_{14}**	**G_{15}**	**G_{16}**
G_{11}	0	1	1	1	0	0
G_{12}	0	0	0	0	0	0
G_{13}	0	0	0	0	0	0
G_{14}	0	0	0	0	0	0
G_{15}	1	1	1	1	0	0
G_{16}	1	1	1	1	0	0

R_5^{G1}	**G_{11}**	**G_{12}**	**G_{13}**	**G_{14}**	**G_{15}**	**G_{16}**
G_{11}	0	1	1	1	1	1
G_{12}	0	0	1	1	1	1
G_{13}	0	0	0	0	0	0
G_{14}	0	0	1	0	1	0
G_{15}	0	0	0	0	0	0
G_{16}	0	0	1	0	1	0

R_6^{G1}	**G_{11}**	**G_{12}**	**G_{13}**	**G_{14}**	**G_{15}**	**G_{16}**
G_{11}	0	0	0	1	1	1
G_{12}	1	0	1	1	1	1
G_{13}	1	0	0	1	1	1
G_{14}	0	0	0	0	0	0
G_{15}	0	0	0	1	0	0
G_{16}	0	0	0	1	0	0

Matrices for G_2:

R_1^{G2}	**G_{21}**	**G_{22}**	**G_{23}**	**G_{24}**
G_{21}	0	1	0	0
G_{22}	0	0	0	0
G_{23}	1	1	0	0
G_{24}	1	1	0	0

R_2^{G2}	**G_{21}**	**G_{22}**	**G_{23}**	**G_{24}**
G_{21}	0	0	0	0
G_{22}	1	0	1	0
G_{23}	0	0	0	0
G_{24}	1	0	1	0

R_3^{G2}	**G_{21}**	**G_{22}**	**G_{23}**	**G_{24}**
G_{21}	0	0	1	0
G_{22}	1	0	1	1
G_{23}	0	0	0	0
G_{24}	0	0	1	0

R_4^{G2}	**G_{21}**	**G_{22}**	**G_{23}**	**G_{24}**
G_{21}	0	1	1	0
G_{22}	0	0	1	0
G_{23}	0	0	0	0
G_{24}	0	1	1	0

R_5^{G2}	**G_{21}**	**G_{22}**	**G_{23}**	**G_{24}**
G_{21}	0	0	1	0
G_{22}	1	0	1	1
G_{23}	0	0	0	0
G_{24}	1	0	1	0

R_6^{G2}	**G_{21}**	**G_{22}**	**G_{23}**	**G_{24}**
G_{21}	0	0	0	0
G_{22}	1	0	1	1
G_{23}	1	0	0	0
G_{24}	1	0	0	0

Matrices for G_3:

R_1^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0	1	1	0	0	0
G_{32}	0	0	0	0	0	0
G_{33}	0	0	0	0	0	0
G_{34}	1	1	1	0	0	0

G_{35}	1	1	1	1	0	1
G_{36}	1	1	1	0	0	0

R_2^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0	0	1	1	1	1
G_{32}	0	0	1	1	1	1
G_{33}	0	0	0	1	0	1
G_{34}	0	0	0	0	0	0
G_{35}	0	0	0	1	0	1
G_{36}	0	0	0	0	0	0

R_3^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0	0	1	1	1	1
G_{32}	0	0	1	1	1	1
G_{33}	0	0	0	1	0	1
G_{34}	0	0	0	0	0	0
G_{35}	0	0	0	1	0	1
G_{36}	0	0	0	0	0	0

R_4^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0	1	1	0	0	1
G_{32}	0	0	1	0	0	1
G_{33}	0	0	0	0	0	0
G_{34}	0	1	1	0	0	1
G_{35}	0	1	1	0	0	1
G_{36}	0	0	0	0	0	0

R_5^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0	1	1	1	1	1
G_{32}	0	0	1	1	1	1
G_{33}	0	0	0	1	1	1
G_{34}	0	0	0	0	0	0
G_{35}	0	0	0	1	0	0
G_{36}	0	0	0	1	1	0

R_6^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0	0	0	0	0	0
G_{32}	1	0	0	0	0	0
G_{33}	1	1	0	0	0	1
G_{34}	1	1	0	0	0	1
G_{35}	1	1	1	1	0	1
G_{36}	1	0	0	0	0	0

The extent to which expert *k* is against G_j is given by the formula:

$$h_j^k = \frac{1}{n-1}\sum_{k=1}^{n} h_{ij}^k, k = 1,...,5; j = 1,....,6$$

This yielded the following matrices:

***j*=**	**G_1**	**G_2**	**G_3**
h_j^1	1/2	0	1
h_j^2	1/2	0	1/2
h_j^3	0	0	1
h_j^4	0	0	0
h_j^5	1	1/2	0
h_j^6	0	1/2	1

Matrix for G_1:

***j*=**	**G_1**	**G_2**	**G_3**	**G_4**	**G_5**	**G_6**
h_j^1	1/5	0	4/5	3/5	1/5	1
h_j^2	2/5	1/5	0	2/5	4/5	0
h_j^3	0	1/5	4/5	2/5	4/5	2/5
h_j^4	2/5	3/5	3/5	3/5	0	0
h_j^5	0	1/5	4/5	2/5	4/5	2/5
h_j^6	2/5	0	1/5	1	3/5	3/5

Matrix for G_2:

***j*=**	**G_1**	**G_2**	**G_3**	**G_4**
h_j^1	2/3	1	0	0
h_j^2	2/3	0	2/3	0
h_j^3	1/3	0	1	1/3
h_j^4	0	2/3	1	0
h_j^5	2/3	0	1	1/3
h_j^6	1	0	1/3	1/3

Matrix for G_3:

***j*=**	**G_1**	**G_2**	**G_3**	**G_4**	**G_5**	**G_6**
h_j^1	3/5	4/5	4/5	1/5	0	1/5
h_j^2	0	0	2/5	4/5	2/5	4/5
h_j^3	0	0	2/5	4/5	2/5	4/5
h_j^4	0	3/5	4/5	0	0	4/5
h_j^5	0	1/5	2/5	1	4/5	3/5
h_j^6	1	3/5	1/5	1/5	0	3/5

The extent to which all individuals are against G_j is given by the formula:

$$h_j^k = \frac{1}{n-1}\sum_{k=1}^{n-1} h_j^k, j = 1,\ldots,6.$$

Yielding the following matrices:

j=	**G_1**	**G_2**	**G_3**
h_j	0.33	0.17	0.58

Matrix for G_1:

j=	**G_1**	**G_2**	**G_3**	**G_4**	**G_5**	**G_6**
h_j	0.23	0.20	0.53	0.57	0.53	0.40

Matrix for G_2:

j=	**G_1**	**G_2**	**G_3**	**G_4**
h_j	0.56	0.28	0.67	0.17

Matrix for G_3:

j=	**G_1**	**G_2**	**G_3**	**G_4**	**G_5**	**G_6**
h_j	0.27	0.37	0.50	0.50	0.27	0.63

From this matrix we can generate the following equation to model U.S. economic growth:

$$P_c = (0.33)G_1 + (0.17)G_2 + (0.58)G_3$$

$$P_c^{G1} = (.23)G_{11} + (.20)G_{12} + (.53)G_{13} + (.57)G_{14} + (.53)G_{15} + (.40)G_{16}$$

$$P_c^{G2} = (.56)G_{21} + (.28)G_{22} + (.67)G_{23} + (.17)G_{24}$$

$$P_c^{G3} = (.27)G_{31} + (.37)G_{32} + (.50)G_{33} + (.50)G_{34} + (.27)G_{35} + (.63)G_{36}$$

Through substitution we can yield the following equation:

$$P_c = (0.33)[(.23)G_{11} + (.20)G_{12} + (.53)G_{13} + (.57)G_{14} + (.53)G_{15} + (.40)G_{16}] + (0.17)[(.56)G_{21} + (.28)G_{22} + (.67)G_{23} + (.17)G_{24}] + (0.58)[(.27)G_{31} + (.37)G_{32} + (.50)G_{33} + (.50)G_{34} + (.27)G_{35} + (.63)G_{36}]$$

The following matrices are generated from this equation:

$$\mu_Q(x) = \begin{cases} 1\ if\ \ x \geq .8, \\ 2x - .6\ if\ \ .3 < x < .8 \\ 0\ if\ \ x \leq .3 \end{cases}$$

R_1^G	G_{11}	G_{12}	G_{13}
G_{11}	0.400	0.483	0.067
G_{12}	0.317	0.400	0.000
G_{13}	0.733	0.817	0.400

R_2^G	G_{11}	G_{12}	G_{13}
G_{11}	0.400	0.471	0.400
G_{12}	0.329	0.400	0.329
G_{13}	0.400	0.471	0.400

R_3^G	G_{11}	G_{12}	G_{13}
G_{11}	0.400	0.400	0.323
G_{12}	0.400	0.400	0.323
G_{13}	0.477	0.477	0.400

R_4^{G1}	G_{11}	G_{12}	G_{13}
G_{11}	0.400	0.400	0.400
G_{12}	0.400	0.400	0.400
G_{13}	0.400	0.400	0.400

R_5^G	G_{11}	G_{12}	G_{13}
G_{11}	0.400	0.733	1.000
G_{12}	0.067	0.400	0.733
G_{13}	0.000	0.067	0.400

R_6^G	G_{11}	G_{12}	G_{13}
G_{11}	0.400	0.313	0.139
G_{12}	0.487	0.400	0.226
G_{13}	0.661	0.574	0.400

Matrices for G_1:

R_1^{G1}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}
G_{11}	0.400	0.441	0.318	0.359	0.400	0.278
G_{12}	0.359	0.400	0.278	0.318	0.359	0.237
G_{13}	0.482	0.522	0.400	0.441	0.482	0.359
G_{14}	0.441	0.482	0.359	0.400	0.441	0.318
G_{15}	0.400	0.441	0.318	0.359	0.400	0.278
G_{16}	0.522	0.563	0.441	0.482	0.522	0.400

R_2^{G1}	$\mathbf{G_{11}}$	$\mathbf{G_{12}}$	$\mathbf{G_{13}}$	$\mathbf{G_{14}}$	$\mathbf{G_{15}}$	$\mathbf{G_{16}}$
$\mathbf{G_{11}}$	0.400	0.456	NA	0.400	0.289	0.511
$\mathbf{G_{12}}$	0.344	0.400	NA	0.344	0.233	0.456
$\mathbf{G_{13}}$	NA	NA	NA	NA	NA	NA
$\mathbf{G_{14}}$	0.400	0.456	NA	0.400	0.289	0.511
$\mathbf{G_{15}}$	0.511	0.567	NA	0.511	0.400	0.622
$\mathbf{G_{16}}$	0.289	0.344	NA	0.289	0.178	0.400

R_3^{G1}	$\mathbf{G_{11}}$	$\mathbf{G_{12}}$	$\mathbf{G_{13}}$	$\mathbf{G_{14}}$	$\mathbf{G_{15}}$	$\mathbf{G_{16}}$
$\mathbf{G_{11}}$	0.400	0.353	0.121	0.307	0.121	0.307
$\mathbf{G_{12}}$	0.447	0.400	0.167	0.353	0.167	0.353
$\mathbf{G_{13}}$	0.679	0.633	0.400	0.586	0.400	0.586
$\mathbf{G_{14}}$	0.493	0.447	0.214	0.400	0.214	0.400
$\mathbf{G_{15}}$	0.679	0.633	0.400	0.586	0.400	0.586
$\mathbf{G_{16}}$	0.493	0.447	0.214	0.400	0.214	0.400

R_4^{G1}	$\mathbf{G_{11}}$	$\mathbf{G_{12}}$	$\mathbf{G_{13}}$	$\mathbf{G_{14}}$	$\mathbf{G_{15}}$	$\mathbf{G_{16}}$
$\mathbf{G_{11}}$	0.400	0.359	0.359	0.359	0.482	0.482
$\mathbf{G_{12}}$	0.441	0.400	0.400	0.400	0.522	0.522
$\mathbf{G_{13}}$	0.441	0.400	0.400	0.400	0.522	0.522
$\mathbf{G_{14}}$	0.441	0.400	0.400	0.400	0.522	0.522
$\mathbf{G_{15}}$	0.318	0.278	0.278	0.278	0.400	0.400
$\mathbf{G_{16}}$	0.318	0.278	0.278	0.278	0.400	0.400

R_5^{G1}	$\mathbf{G_{11}}$	$\mathbf{G_{12}}$	$\mathbf{G_{13}}$	$\mathbf{G_{14}}$	$\mathbf{G_{15}}$	$\mathbf{G_{16}}$
$\mathbf{G_{11}}$	0.400	0.267	0.000	0.067	0.000	0.067
$\mathbf{G_{12}}$	0.533	0.400	0.000	0.200	0.000	0.200
$\mathbf{G_{13}}$	1.000	0.867	0.400	0.667	0.400	0.667
$\mathbf{G_{14}}$	0.733	0.600	0.133	0.400	0.133	0.400
$\mathbf{G_{15}}$	1.000	0.867	0.400	0.667	0.400	0.667
$\mathbf{G_{16}}$	0.733	0.600	0.133	0.400	0.133	0.400

R_6^{G1}	$\mathbf{G_{11}}$	$\mathbf{G_{12}}$	$\mathbf{G_{13}}$	$\mathbf{G_{14}}$	$\mathbf{G_{15}}$	$\mathbf{G_{16}}$
$\mathbf{G_{11}}$	0.400	0.489	0.444	0.222	0.356	0.356
$\mathbf{G_{12}}$	0.311	0.400	0.356	0.133	0.267	0.267
$\mathbf{G_{13}}$	0.356	0.444	0.400	0.178	0.311	0.311
$\mathbf{G_{14}}$	0.578	0.667	0.622	0.400	0.533	0.533
$\mathbf{G_{15}}$	0.444	0.533	0.489	0.267	0.400	0.400
$\mathbf{G_{16}}$	0.444	0.533	0.489	0.267	0.400	0.400

Matrices for G_2:

R_1^{G2}	$\mathbf{G_{21}}$	$\mathbf{G_{22}}$	$\mathbf{G_{23}}$	$\mathbf{G_{24}}$
$\mathbf{G_{21}}$	0.400	0.289	0.456	0.456

	G_{21}	G_{22}	G_{23}	G_{24}
G_{22}	0.511	0.400	0.567	0.567
G_{23}	0.344	0.233	0.400	0.400
G_{24}	0.344	0.233	0.400	0.400

R_2^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0.400	0.511	0.400	0.511
G_{22}	0.289	0.400	0.289	0.400
G_{23}	0.400	0.511	0.400	0.511
G_{24}	0.289	0.400	0.289	0.400

R_3^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0.400	0.456	0.344	0.400
G_{22}	0.344	0.400	0.289	0.344
G_{23}	0.456	0.511	0.400	0.456
G_{24}	0.400	0.456	0.344	0.400

R_4^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0.400	0.331	0.262	0.400
G_{22}	0.469	0.400	0.331	0.469
G_{23}	0.538	0.469	0.400	0.538
G_{24}	0.400	0.331	0.262	0.400

R_5^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0.400	0.757	0.329	0.686
G_{22}	0.043	0.400	0.000	0.329
G_{23}	0.471	0.829	0.400	0.757
G_{24}	0.114	0.471	0.043	0.400

R_6^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0.400	0.543	0.471	0.471
G_{22}	0.257	0.400	0.329	0.329
G_{23}	0.329	0.471	0.400	0.400
G_{24}	0.329	0.471	0.400	0.400

Matrices for G_3:

R_1^{G3}	G_{31}	G_{32}	G_{33}	G_{34}	G_{35}	G_{36}
G_{31}	0.400	0.331	0.331	0.538	0.607	0.538
G_{32}	0.469	0.400	0.400	0.607	0.676	0.607
G_{33}	0.469	0.400	0.400	0.607	0.676	0.607
G_{34}	0.262	0.193	0.193	0.400	0.469	0.400
G_{35}	0.193	0.124	0.124	0.331	0.400	0.331
G_{36}	0.262	0.193	0.193	0.400	0.469	0.400

R_2^{G3}	G_{31}	G_{32}	G_{33}	G_{34}	G_{35}	G_{36}
G_{31}	0.400	0.400	0.362	0.285	0.362	0.285

G_{32}	0.400	0.400	0.362	0.285	0.362	0.285
G_{33}	0.438	0.438	0.400	0.323	0.400	0.323
G_{34}	0.515	0.515	0.477	0.400	0.477	0.400
G_{35}	0.438	0.438	0.400	0.323	0.400	0.323
G_{36}	0.515	0.515	0.477	0.400	0.477	0.400

R_3^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0.400	0.400	0.362	0.285	0.362	0.285
G_{32}	0.400	0.400	0.362	0.285	0.362	0.285
G_{33}	0.438	0.438	0.400	0.323	0.400	0.323
G_{34}	0.515	0.515	0.477	0.400	0.477	0.400
G_{35}	0.438	0.438	0.400	0.323	0.400	0.323
G_{36}	0.515	0.515	0.477	0.400	0.477	0.400

R_4^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0.400	0.300	0.250	0.400	0.400	0.250
G_{32}	0.500	0.400	0.350	0.500	0.500	0.350
G_{33}	0.550	0.450	0.400	0.550	0.550	0.400
G_{34}	0.400	0.300	0.250	0.400	0.400	0.250
G_{35}	0.400	0.300	0.250	0.400	0.400	0.250
G_{36}	0.550	0.450	0.400	0.550	0.550	0.400

R_5^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0.400	0.000	0.000	0.000	0.000	0.000
G_{32}	0.800	0.400	0.320	0.080	0.160	0.240
G_{33}	0.880	0.480	0.400	0.160	0.240	0.320
G_{34}	1.000	0.720	0.640	0.400	0.480	0.560
G_{35}	1.000	0.640	0.560	0.320	0.400	0.480
G_{36}	0.960	0.560	0.480	0.240	0.320	0.400

R_6^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0.400	0.554	0.605	0.605	0.759	0.554
G_{32}	0.246	0.400	0.451	0.451	0.605	0.400
G_{33}	0.195	0.349	0.400	0.400	0.554	0.349
G_{34}	0.195	0.349	0.400	0.400	0.554	0.349
G_{35}	0.041	0.195	0.246	0.246	0.400	0.195
G_{36}	0.246	0.400	0.451	0.451	0.605	0.400

This data is the core set of dominate goals, those that are not defeated by the pairwise comparisons by a majority. The following will determine the secondary core:

$$h_{ij}^{k} = \begin{cases} 1\, if\ r_{ij}^{k} > .5, \\ 0\ otherwise \end{cases}$$

Giving the following matrices where R_i denotes the expert:

R_1	G_1	G_2	G_3
G_1	0	1	0
G_2	0	0	0
G_3	1	1	0

R_2	G_1	G_2	G_3
G_1	0	1	0
G_2	0	0	0
G_3	0	1	0

R_3	G_1	G_2	G_3
G_1	0	0	0
G_2	0	0	0
G_3	1	1	0

R_4	G_1	G_2	G_3
G_1	0	0	0
G_2	0	0	0
G_3	0	0	0

R_5	G_1	G_2	G_3
G_1	0	1	1
G_2	0	0	1
G_3	0	0	0

R_6	G_1	G_2	G_3
G_1	0	0	0
G_2	1	0	0
G_3	1	1	0

This process was the repeated for the sub factors, G_{ij}, for each of the G_i's. Matrices for G_1:

R_1^{G1}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}
G_{11}	0	1	0	0	0	0
G_{12}	0	0	0	0	0	0
G_{13}	1	1	0	1	1	0
G_{14}	1	1	0	0	1	0
G_{15}	0	1	0	0	0	0
G_{16}	1	1	1	1	1	0

R_2^{G1}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}
G_{11}	0	1	1	0	0	1
G_{12}	0	0	1	0	0	1

G_{13}	1	1	1	1	1	1
G_{14}	0	1	1	0	0	1
G_{15}	1	1	1	1	0	1
G_{16}	0	0	1	0	0	0

R_3^{G1}	**G_{11}**	**G_{12}**	**G_{13}**	**G_{14}**	**G_{15}**	**G_{16}**
G_{11}	0	0	0	0	0	0
G_{12}	1	0	0	0	0	0
G_{13}	1	1	0	1	0	1
G_{14}	1	1	0	0	0	0
G_{15}	1	1	0	1	0	1
G_{16}	1	1	0	0	0	0

R_4^{G1}	**G_{11}**	**G_{12}**	**G_{13}**	**G_{14}**	**G_{15}**	**G_{16}**
G_{11}	0	0	0	0	1	1
G_{12}	1	0	0	0	1	1
G_{13}	1	0	0	0	1	1
G_{14}	1	0	0	0	1	1
G_{15}	0	0	0	0	0	0
G_{16}	0	0	0	0	0	0

R_5^{G1}	**G_{11}**	**G_{12}**	**G_{13}**	**G_{14}**	**G_{15}**	**G_{16}**
G_{11}	0	0	0	0	0	0
G_{12}	1	0	0	0	0	0
G_{13}	1	1	0	1	0	1
G_{14}	1	1	0	0	0	0
G_{15}	1	1	0	1	0	1
G_{16}	1	1	0	0	0	0

R_6^{G1}	**G_{11}**	**G_{12}**	**G_{13}**	**G_{14}**	**G_{15}**	**G_{16}**
G_{11}	0	1	1	0	0	0
G_{12}	0	0	0	0	0	0
G_{13}	0	1	0	0	0	0
G_{14}	1	1	1	0	1	1
G_{15}	1	1	1	0	0	0
G_{16}	1	1	1	0	0	0

Matrices for G_2:

R_1^{G2}	**G_{21}**	**G_{22}**	**G_{23}**	**G_{24}**
G_{21}	0	0	1	1
G_{22}	1	0	1	1
G_{23}	0	0	0	0
G_{24}	0	0	0	0

R_2^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0	1	0	1
G_{22}	0	0	0	0
G_{23}	0	1	0	1
G_{24}	0	0	0	0

R_3^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0	1	0	0
G_{22}	0	0	0	0
G_{23}	1	1	0	1
G_{24}	0	1	0	0

R_4^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0	0	0	0
G_{22}	1	0	0	1
G_{23}	1	1	0	1
G_{24}	0	0	0	0

R_5^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0	1	0	1
G_{22}	0	0	0	0
G_{23}	1	1	0	1
G_{24}	0	1	0	0

R_6^{G2}	G_{21}	G_{22}	G_{23}	G_{24}
G_{21}	0	1	1	1
G_{22}	0	0	0	0
G_{23}	0	1	0	0
G_{24}	0	1	0	0

Matrices for G_3:

R_1^{G3}	G_{31}	G_{32}	G_{33}	G_{34}	G_{35}	G_{36}
G_{31}	0	0	0	1	1	1
G_{32}	1	0	0	1	1	1
G_{33}	1	0	0	1	1	1
G_{34}	0	0	0	0	1	0
G_{35}	0	0	0	0	0	0
G_{36}	0	0	0	0	1	0

R_2^{G3}	G_{31}	G_{32}	G_{33}	G_{34}	G_{35}	G_{36}
G_{31}	0	0	0	0	0	0
G_{32}	0	0	0	0	0	0
G_{33}	1	1	0	0	0	0
G_{34}	1	1	1	0	1	0

G_{35}	1	1	0	0	0	0
G_{36}	1	1	1	0	1	0

R_3^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0	0	0	0	0	0
G_{32}	0	0	0	0	0	0
G_{33}	1	1	0	0	0	0
G_{34}	1	1	1	0	1	0
G_{35}	1	1	0	0	0	0
G_{36}	1	1	1	0	1	0

R_4^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0	0	0	0	0	0
G_{32}	1	0	0	1	1	0
G_{33}	1	1	0	1	1	0
G_{34}	0	0	0	0	0	0
G_{35}	0	0	0	0	0	0
G_{36}	1	1	0	1	1	0

R_5^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0	0	0	0	0	0
G_{32}	1	0	0	0	0	0
G_{33}	1	1	0	0	0	0
G_{34}	1	1	1	0	1	1
G_{35}	1	1	1	0	0	1
G_{36}	1	1	1	0	0	0

R_6^{G3}	**G_{31}**	**G_{32}**	**G_{33}**	**G_{34}**	**G_{35}**	**G_{36}**
G_{31}	0	1	1	1	1	1
G_{32}	0	0	1	1	1	0
G_{33}	0	0	0	0	1	0
G_{34}	0	0	0	0	1	0
G_{35}	0	0	0	0	0	0
G_{36}	0	0	1	1	1	0

The extent to which expert *k* is against G_j is given by the formula:

$$h_j^k = \frac{1}{n-1}\sum_{k=1}^{n} h_{ij}^k, k = 1,...,5; j = 1,....,6$$

This yielded the following matrices:

***j*=**	**G_1**	**G_2**	**G_3**

h_j^1	1/2	1	0
h_j^2	0	1	0
h_j^3	1/2	1/2	0
h_j^4	0	0	0
h_j^5	0	1/2	1
h_j^6	1	1/2	0

Matrix for G_1:

j=	**G_1**	**G_2**	**G_3**	**G_4**	**G_5**	**G_6**
h_j^1	3/5	1	1/5	2/5	3/5	0
h_j^2	2/5	4/5	1 1/5	2/5	1/5	1
h_j^3	1	4/5	0	2/5	0	2/5
h_j^4	3/5	0	0	0	4/5	4/5
h_j^5	1	4/5	0	2/5	0	2/5
h_j^6	3/5	1	4/5	0	1/5	1/5

Matrix for G_2:

j=	**G_1**	**G_2**	**G_3**	**G_4**
h_j^1	1/3	0	2/3	2/3
h_j^2	0	2/3	0	2/3
h_j^3	1/3	1	0	1/3
h_j^4	2/3	1/3	0	2/3
h_j^5	1/3	1	0	2/3
h_j^6	0	1	1/3	1/3

Matrix for G_3:

j=	**G_1**	**G_2**	**G_3**	**G_4**	**G_5**	**G_6**
h_j^1	2/5	0	0	3/5	1	3/5
h_j^2	4/5	4/5	2/5	0	2/5	0
h_j^3	4/5	4/5	2/5	0	2/5	0
h_j^4	3/5	2/5	0	3/5	3/5	0
h_j^5	1	4/5	3/5	0	1/5	2/5
h_j^6	0	1/5	3/5	3/5	1	1/5

The extent to which all individuals are in favor G_j is given by the formula:

$$h_j^k = \frac{1}{n-1}\sum_{k=1}^{n-1} h_j^k, j = 1,\ldots,6.$$

Yielding the following matrices:

***j*=**	G_1	G_2	G_3
h_j	0.33	0.58	0.17

Matrix for G_1:

***j*=**	G_1	G_2	G_3	G_4	G_5	G_6
h_j	0.70	0.73	0.37	0.27	0.30	0.47

Matrix for G_2:

***j*=**	G_1	G_2	G_3	G_4
h_j	0.28	0.67	0.17	0.56

Matrix for G_3:

***j*=**	G_1	G_2	G_3	G_4	G_5	G_6
h_j	0.60	0.50	0.33	0.30	0.60	0.20

From this matrix we can generate the following equations the expert's favor toward each goal and its affect on U.S. economic growth:

$$P_f= (0.33)G_1 + (0.58)G_2 + (0.17)G_3$$

$$P_f^{G1}= (.70)G_{11} + (.73)G_{12} + (.37)G_{13} + (.27)G_{14} + (.30)G_{15} + (.47)G_{16}$$

$$P_f^{G2}= (.28)G_{21} + (.67)G_{22} + (.17)G_{23} + (.56)G_{24}$$

$$P_f^{G3}= (.60)G_{31} +(.50)G_{32} + (.33)G_{33} + (.30)G_{34} + (.60)G_{35} + (.20)G_{36}$$

Through substitution we can yield the following equation:

$$P_f= (0.33)[(.70)G_{11} + (.73)G_{12} + (.37)G_{13} + (.27)G_{14} + (.30)G_{15} + (.47)G_{16}] + (0.58)[(.28)G_{21} + (.67)G_{22} + (.17)G_{23} + (.56)G_{24}] + (0.17)[(.60)G_{31} +(.50)G_{32} + (.33)G_{33} + (.30)G_{34} + (.60)G_{35} + (.20)G_{36}]$$

5.1 Conclusions

Through a fuzzy mathematical analysis of U.S. economic growth varying models can be determined. The following equation was determined to model economic growth:

$$P= (.322)[(.214)G_{11} + (.211)G_{12} + (.122)G_{13} + (.157)G_{14} + (.135)G_{15}+ (.181)G_{16}] + (.356)[(.232)G_{21} + (.272)G_{22} + (.220)G_{23} + (.275)G_{24}] + (.322)[(.196)G_{31} + (.162)G_{32} + (.155)G_{33} + (.152)G_{34} + (.187)G_{35} + (.149)G_{36}]$$

From this equation the expert's unfavorability toward each goal was modeled into the following equation to show the strengths of the opinions against each issue.

$$P_c = (0.33)[(.23)G_{11} + (.20)G_{12} + (.53)G_{13} + (.57)G_{14} + (.53)G_{15} + (.40)G_{16}] + (0.17)[(.56)G_{21} + (.28)G_{22} + (.67)G_{23} + (.17)G_{24}] + (0.58)[(.27)G_{31} + (.37)G_{32} + (.50)G_{33} + (.50)G_{34} + (.27)G_{35} + (.63)G_{36}]$$

A model for the favorability of each sub goal in its effect on economic growth were then modeled into the following equation.

$$P_f = (0.33)[(.70)G_{11} + (.73)G_{12} + (.37)G_{13} + (.27)G_{14} + (.30)G_{15} + (.47)G_{16}] + (0.58)[(.28)G_{21} + (.67)G_{22} + (.17)G_{23} + (.56)G_{24}] + (0.17)[(.60)G_{31} + (.50)G_{32} + (.33)G_{33} + (.30)G_{34} + (.60)G_{35} + (.20)G_{36}]$$

From expert opinion these models can be used to model and predict U.S. economic growth. These models can be developed further to incorporate data from past years and apply the data to this model. This will test the efficiency of the model by predicting past years economic growth. If the equation models the data correctly it can be seen that the model is effective for predicting future years economic growth.

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The Semiotics of Self-Identity and the Danger of Mirror Stage Simulation

Natalie Gorup

Creighton University

From observing a child's fascination with his or her reflection, one would never guess the danger imposed by a simple sheet of silvered-glass. But when we consider the reflected image with regards to Ferdinand de Saussure's notion of sign along with the implications of Jacques Lacan's mirror stage and Jean Baudrillard's simulations, we can uncover the threat it poses to something that concerns us all: self-identity. While Lacan and Baudrillard chiefly explore self-identity and sign (respectively) in relation to society, I would like to apply their insights to the common conundrum of self-identity, ignoring, for the present moment, societal influences. I would also like to assert that despite Lacan's limitation of his mirror stage to a child's first few years, the human interaction with self-image in the form of reflection continues to impact self identity beyond this short window. As I will show, the simulation of our reflections jeopardizes our self-identities. Understanding the danger our reflections pose to our ability to fulfill the universal need to, as the ancient Greeks put it, "know thyself," will change the way we think about our reflections and potentially allow us to better know and protect our true Selves.

Self-identity, in Lacan's view, is largely shaped by the effects of seeing our "self-image" as a reflection in the mirror, which he designates as occurring at anywhere from six to eighteen months (442). Lacan sees this mirror stage as "an *identification* . . . the transformation that takes place in the subject when he assumes the image. . . ” (442). He “regard[s] the function of the mirror stage. . . to establish a relation between the organism and its reality” (443). According to Saussure’s notion of sign (as composed of signifier and signified, 62), the image in the mirror acts as signifier for the Self, the signified, which it reflects. Thus the connection between self-image and self-identity can be explored in terms of semiotics.

While Lacan explores the impact this perception of self (through reflection) has upon the young human’s self-identity in relation to society, in light of Jean Baudrillard’s work on simulation, Lacan’s observations remain important even when focusing on the self alone and temporarily ignoring its relation to the rest of society. To Baudrillard, the danger of sign is the threat of simulation, in which reality is endangered by the disappearance of the signified behind the signifier and the assumption of the signifier as “real” (366). When considered together, Lacan’s and Baudrillard’s observations illustrate the pivotal link between self-image and self-identity, and suggest the danger of allowing the former to replace the latter.

I personally discovered the danger of simulation after the typical Lacanian mirror stage during an “experiment” of sorts which I performed. During one season of Lent, (in which Catholics sacrifice something in order to grow closer to God), I chose to give up looking at my reflection in the mirror. I covered my bathroom mirror with wrapping paper, and kept my eyes downcast when passing mirrors in public places. While initially motivated out of an attempt to avoid vanity and make better use of my time, this spiritual exercise unnerved me more and more as it turned into a sort of self- identity crisis. I would have never guessed that my Lenten sacrifice would threaten my sense of self-identity. However, although I didn't want to admit it, the reflection that was initially only a signifier for my true Self, the signified, had threatened to replace my true Self as the "real" in my mind. It was frightening to realize how much I based my sense of self on my reflection, a mere signifier. I had allowed my perception of "who I was" to be defined by the reflection I saw in the mirror every morning. I "was" my appearance, instead of my inner self that actually caused change and produced effects in the world. Instead of looking *into* my Self, I had been looking *at* my self (small *s*), my image, and the effects were startling.

During my Lenten journey, as I turned away from the outward to the inward, I returned to the signified from the signifier. The importance of returning to the origin of my self-identity was made obvious by the very function of sign: The signifier stands for the signified that is not fully present; it is representative (366). But I, being fully present, did not need my image to stand for my presence as it might in a photograph sent to a far-away loved one. Unknowingly, I had let my image replace my Self in my mind instead of letting it simply *stand* for me, as a self with a small *s*. Once I had recognized the danger of allowing the subject "I" to be reduced to the object "me," I knew I had to relearn that I was more than what I saw in the mirror.

This objectification of Self explains the time many spend before the mirror improving their outward appearance: Those who expend time and energy on such "self-improvement" have fallen prey to the replacement of the self with the Self. Their actions suggest, (to utilize a Freudian perspective), a perhaps unconscious belief that by improving their images, the signifiers, they can improve the signifieds, their true Selves. What they do not realize is that the "I" which existed for each of them before they saw their reflections is the true Self; it cannot be changed simply by changing the signifiers, their images.

Ignoring the practical use of mirrors to make oneself "presentable," the mirror has no other function—unless you are a narcissist, in which case the Self is lost completely to that self (small *s*) depicted by the reflection. The tale of Narcissus can therefore be reinterpreted not as a warning against Self-admiration, but of self-admiration (small *s*)-- of an obsession with the signifier, and a neglect of our true Selves, the signified. Thus the link between the Self and the self exposes the reflection as a potential threat to self-identity.

Just as, through my mirror exercise, I had to relearn that the "me" I saw in the mirror was separate from and inferior to the "I" it reflected, so must we all separate between self-image and self-identity. During the course of my exercise, I was astounded by the sheer number of mirrors fixed in public places, reminiscent of Baudrillard's fear that "reality [is] being increasingly replaced by sign systems that. . . replace the real" (365). He expressed his fear in the Borges tale, in which the detailed map of an empire becomes so close to reality that it ends up replacing the "real" empire (365). When we understand our reflections as signifiers, we must share Baudrillard's fear that "an aging double [will] end[s] up being confused with the real thing" (365) and the real thing itself will vanish. In our terms of mirrors and self-images, we must fear that, should we focus too much on our reflections, we could end up forgetting the true Selves being reflected. Although it is a radical assertion, if we follow Baudrillard's model, we are brought to the startling conclusion that, over time, as the signifieds began to disappear, so too would their reflections; for without a Self to reflect, the inferior self (small *s*) could not exist. Only then would we realize the "murderous capacity of images" (368) and thus our fatal mistake; by then it would then be too late.

To avoid Baudrillard's fear of an "age of simulation" (366) beginning with a "liquidation of all referentials" (366) in which signifiers are substituted for the real, we must guard against the shift from representation to simulation. Here, instead of representing the signified, the signifier becomes the "death sentence of every reference" (368). We must ensure that we do not make the move from signifier as a *reflection* of a basic reality to a *replacement* of reality (368), for if we remember Saussure's assertion of the Arbitrary Nature of Signs (62), we will realize that reflections and self-images (as signifiers) are not, in fact, intrinsically linked to our true Selves, the signifieds. It is therefore imperative that we preserve what it is the signifier stands for, recognizing and reacting to the danger of dependence on signifiers as replacements for the real. So let us preserve the reality of the signified; let us step away from the mirror for a while, and reacquaint ourselves with the signified, our true Selves.

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A New Notion of Maximality for Fuzzy Preference Spatial Models
John E. Herr

Introduction

This paper explores a new method of applying fuzzy mathematics to spatial models of social choice theory. Given a set of alternatives, the assumption that an individual has a *single-peaked profile* means that he has a "unique most preferred alternative", which especially in a policy space is called an *ideal point* (Austen-Smith & Banks 93). Certain fuzzy approaches suggest that we should not ask which alternative is the player's ideal point, but rather to what degree each alternative is ideal. Thus the preferences of an individual are represented by the assignment of a fuzzy number to each alternative.

The fuzzy approach taken by Clark, Larson, Mordeson, Potter, and Wierman (2008) is to use the varying degrees to which an alternative is considered ideal by a player to partition the set of alternatives X into finitely many regions, thus inducing a binary preference relation R on X, where the player is indifferent to alternatives contained in the same region. An aggregation rule may often be used to take the set of preference relations of a group of individuals and return a single preference relation, interpreted as their collective preference. If R is a binary preference relation on X, the conventional *maximal set* is defined to be $\{x \in X: \forall y \in X, xRy\}$.

I take an alternative approach that inserts fuzzy mathematics in a different conceptual location. Instead of thinking of the degree to which an alternative is ideal, the initial description of a player's preferences will fuzzily compare each point to one other point at a time. This replaces the exact binary weak preference relation with a preference function, so that instead of saying "x is preferred or indifferent to y," or "xRy," we specify the degree to which x is preferred over y. This is similar to the course taken by Dutta (217), though our preference function Λ need not be considered the fuzzy weak preference relation (FWPR) spoken of by him, because we do not necessarily regard it as a fuzzified version of R. However, it is certainly possible to do so if desired. Since ultimately a social preference must choose affirmatively between alternatives, Dutta notes that "the *fuzzy social preferences* relation has to generate *exact social choice*" (216). This is acknowledged as probably true, but it is not explored in this paper any further than the analysis of a few specific aggregation criterion in conjunction with a new notion of maximality, as shall be seen. The important point is that fuzzy mathematics will be used to describe relative preference, rather than assuming the existence of absolute idealness.

Because a preference function is used rather than an exact binary preference relation, a new notion of a maximal set is defined. This maximal set is fuzzy in the truest sense, and its definition, though simple, is the central idea of this paper. Involved in its definition is integral calculus, which takes advantage of the topological properties of spatial models.

The Type-2 Fuzzy Maximal Set

Let X be a subset of a locally compact metric space equipped with a Radon measure μ such that the μ-measure of X is finite and nonzero.

Define a function $\Lambda: X^2 \to [0,1]$ such that $\Lambda_x: X \to [0,1]$, where $\Lambda_x(y) = \Lambda(x, y)$, is integrable with respect to μ on X for all $x \in X$. We interpret $\Lambda_x(y)$ as the degree to which x is preferred over y. We may refer to such a function Λ as a **preference function**.

Note: It is, in fact, possible to induce a binary relation on X from Λ by choosing $\alpha, \beta \in [0,1], \alpha \leq \beta$, and then defining xRy if and only if $\Lambda_x(y) \geq \alpha$, with xPy if and only if $\Lambda_x(y) > \beta$.

We define the **type-2 fuzzy maximal set** to be the function $\Phi: X \to [0,1]$ defined by

$$\Phi(x) = \frac{\int_X \Lambda_x \, d\mu}{\int_X d\mu}$$

Example: Let $X = [0,1]^2$. Then X is a subset of a locally compact metric space (the Euclidean metric), where we take the Radon measure μ to be the Lebesgue measure in the plane. Suppose we define a utility function $u: X \to \mathbb{R}$ by $u(x,y) = -\sqrt{\left(x-\frac{1}{4}\right)^2 + \left(y-\frac{1}{4}\right)^2}$. In other words, the point $\left(\frac{1}{4},\frac{1}{4}\right)$ is an ideal point, with the utility function assigning to a point the negative of its Euclidean distance from $\left(\frac{1}{4},\frac{1}{4}\right)$.

From the utility function, it is possible to define a preference function $\Lambda: X^2 \to [0,1]$ by

$$\Lambda(x,y) = \frac{1}{e^{u(y)-u(x)} + 1}$$

In fact, in almost any situation, if u is a utility function, we can obtain a valid preference function Λ in this way. However, this does not mean this Λ will be easy to use. In this case, for example, we have

$$\Lambda_{(x,y)}\big((z,w)\big) = \frac{1}{e^{\left(-\sqrt{\left(z-\frac{1}{4}\right)^2+\left(w-\frac{1}{4}\right)^2}+\sqrt{\left(x-\frac{1}{4}\right)^2+\left(y-\frac{1}{4}\right)^2}\right)} + 1}$$

Note that since $X = [0,1]^2$, we have $\int_X \; d\mu = 1$. Hence,

$$\Phi\big((x,y)\big) = \int_X \Lambda_{(x,y)} \, d\mu = \int_0^1 \int_0^1 \frac{1}{e^{\left(-\sqrt{\left(z-\frac{1}{4}\right)^2+\left(w-\frac{1}{4}\right)^2}+\sqrt{\left(x-\frac{1}{4}\right)^2+\left(y-\frac{1}{4}\right)^2}\right)} + 1} \, dw \, dz$$

Regretfully, this integral is nearly impossible to evaluate in closed form, and hence ought to be evaluated numerically. Another possible downside is that a preference function based on a utility function may not be possessed of its own independent meaning.

Instead, let's bypass the utility function u and instead define Λ directly as

$$\Lambda_{(x,y)}\big((z,w)\big) = \frac{2^{z-x} + 2^{w-y}}{4}$$

For $(x, y), (z, w) \in [0,1]^2$, Λ has a range contained in $[0,1]$, as desired. Notice that this particular preference function "prefers" points closer to $(0,0)$. For example, $\Lambda_{(0,0)}\left(\left(\frac{1}{2}, \frac{1}{2}\right)\right) = \frac{\sqrt{2}}{2}$ while $\Lambda_{\left(\frac{1}{2}, \frac{1}{2}\right)}\left((0,0)\right) = \frac{\sqrt{2}}{4}$, meaning $(0,0)$ is preferred over $\left(\frac{1}{2}, \frac{1}{2}\right)$ by degree $\frac{\sqrt{2}}{2}$, whereas $\left(\frac{1}{2}, \frac{1}{2}\right)$ is preferred over $(0,0)$ only by degree $\frac{\sqrt{2}}{4}$.

Now, the type-2 fuzzy maximal set is given by

$$\Phi\left((x, y)\right) = \int_X \Lambda_{(x,y)}\, d\mu = \int_0^1 \int_0^1 \frac{2^{z-x} + 2^{w-y}}{4}\, dw\, dz = \frac{2^{-x} + 2^{-y}}{4 \ln 2}$$

Let's take a look at what the type-2 fuzzy maximal set looks like graphically:

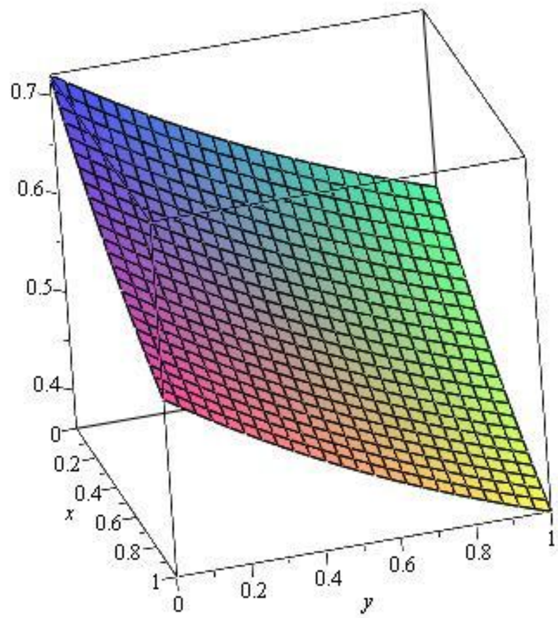

Figure 1: The type-2 fuzzy maximal set defined by $\boldsymbol{\Phi}\left((x, y)\right) = \frac{2^{-x}+2^{-y}}{4 \ln 2}$

As we can see, points closer to $(0,0)$ are in the type-2 fuzzy maximal set to a greater degree than those farther away.

A Brief Exploratory Extension

If X is a set of alternatives and Φ is a type-2 fuzzy maximal set corresponding to X, then every point of X is in Φ to some degree in the range [0,1]. We may wish to restrict our attention to those points that are in Φ to a sufficient degree. Therefore, we have the following definition:

Definition: Let Φ be a type-2 fuzzy maximal set and let $\alpha \in [0,1]$. The **$\boldsymbol{\alpha}$-level set of $\boldsymbol{\Phi}$** is the set $\Phi^{-1}([\alpha, 1])$.

For example, the 1-level set of Φ is the set of all points $x \in X$ such that $\Phi(x) = 1$, i.e. those points that are perfectly members of the type-2 fuzzy maximal set. The α-level set is an established notion in fuzzy mathematics. More generally, if $\tilde{F}$ is a fuzzy set, the α-level set of $\tilde{F}$ is the set $\{x \in X: \tilde{F}(x) \geq \alpha\}$ and is denoted $\tilde{F}^{\alpha}$. However, we will continue to emphasize Φ as a function and will use the corresponding notation.

If there are multiple players considering a set of alternatives X, then each player will have his own preference function over X and hence his own type-2 fuzzy maximal set Φ. We may discover some information about how the players may interact with one another by comparing their type-2 fuzzy maximal sets. We begin this examination with the following definition:

Definition: Let $N = \{1, \dots, k\}$ be a set of players, each player $n \in N$ having his own preference function and corresponding type-2 fuzzy maximal set Φ_n. The **threshold of accord among $\boldsymbol{N}$** is γ_N, where

$$\gamma_N = \sup\left\{r \in [0,1]: \bigcap_{n \in N} \Phi_n^{-1}([r, 1]) \neq \emptyset\right\}$$

In other words, γ_N is the supremum of those $r \in [0,1]$ such that the intersection of all the players' r-level sets is nonempty. This means that if $r \in [0,1]$ with $r < \gamma_N$, then $\bigcap_{n \in N} \Phi_n^{-1}([r, 1]) \neq \emptyset$. It does <u>not</u> mean that $\bigcap_{n \in N} \Phi_n^{-1}([\gamma_N, 1]) \neq \emptyset$. The intersection of the γ_N-

levels may or may not be itself nonempty. The threshold of accord is intuitively a measure of how restrictive the level sets are that we can impose on all the players before they no longer share a common element.

Naturally, our goal is to discover where the players might come to a consensus. One possibility is that they decide to choose a point $x \in X$ such that the misery of the least-satisfied player, in terms of x's membership in the players' type-2 fuzzy maximal sets, is as low as possible. In other words, they may decide to settle on an $x \in X$ such that for all $y \in X$, $\min_{n\in N} \Phi_n(y) \leq \min_{n\in N} \Phi_n(x)$. However, such an x does not have to exist, and if it does not, there is no reason to believe that the players will be able to come to an agreement if this maximization of the least satisfied (MLS) criterion is being used. Fortunately, there is a condition that implies the existence of such an x.

Of course, the set of such x's can be described as $\{x \in X: \forall y \in X, \min_{n\in N} \Phi_n(y) \leq \min_{n\in N} \Phi_n(x)\}$. Through the following lemma, we arrive at a useful conclusion:

Lemma 1: Let $N = \{1, \dots, k\}$ be a set of players with corresponding type-2 fuzzy maximal sets Φ_n and let $R = \{r \in [0,1]: \bigcap_{n\in N} \Phi_n^{-1}([r,1]) \neq \emptyset\}$. Then

$$\bigcap_{r\in R}\left(\bigcap_{n\in N} \Phi_n^{-1}([r,1])\right) \subseteq \left\{x \in X: \forall y \in X, \min_{n\in N} \Phi_n(y) \leq \min_{n\in N} \Phi_n(x)\right\}$$

Proof: Let $x \in \bigcap_{r\in R}(\bigcap_{n\in N} \Phi_n^{-1}([r,1]))$. Assume, for the sake of contradiction, that $x \notin \{x \in X: \forall y \in X, \min_{n\in N} \Phi_n(y) \leq \min_{n\in N} \Phi_n(x)\}$. Then there exists $y \in X$ such that $\min_{n\in N} \Phi_n(y) > \min_{n\in N} \Phi_n(x)$. We have $y \in \Phi_n^{-1}([\min_{n\in N} \Phi_n(y), 1])$ for all $n \in N$. Hence, $y \in \bigcap_{n\in N} \Phi_n^{-1}([\min_{n\in N} \Phi_n(y), 1])$, and it follows that $\min_{n\in N} \Phi_n(y) \in R$. Since $\min_{n\in N} \Phi_n(y) > \min_{n\in N} \Phi_n(x)$, there exists $n \in N$ such that $\Phi_n(x) < \min_{n\in N} \Phi_n(y)$. Hence, $x \notin \bigcap_{n\in N} \Phi_n^{-1}([\min_{n\in N} \Phi_n(y), 1])$. However, this contradicts the fact that

$x \in \bigcap_{n\in N} \Phi_n^{-1}([r,1])$ for all $r \in R$, because $\min_{n\in N} \Phi_n(y) \in R$. Hence, $x \in \{x \in X: \forall y \in X, \min_{n\in N} \Phi_n(y) \leq \min_{n\in N} \Phi_n(x)\}$, and thus $\bigcap_{r\in R}(\bigcap_{n\in N} \Phi_n^{-1}([r,1])) \subseteq \{x \in X: \forall y \in X, \min_{n\in N} \Phi_n(y) \leq \min_{n\in N} \Phi_n(x)\}$. ■

With the help of Lemma 1, we arrive at the following theorem, which gives a condition under which there exist points satisfying the MLS criterion:

Theorem 1: Let $N = \{1, \dots, k\}$ be a set of players with corresponding type-2 fuzzy maximal sets Φ_n. Suppose Φ_n is continuous on X for all $n \in N$. Suppose further that X is compact. Then

$$\left\{x \in X: \forall y \in X, \min_{n\in N} \Phi_n(y) \leq \min_{n\in N} \Phi_n(x)\right\} \neq \emptyset$$

Proof: Let $R = \{r \in [0,1]: \bigcap_{n\in N} \Phi_n^{-1}([r,1]) \neq \emptyset\}$. Since Φ_n is continuous on X for all $n \in N$, $\Phi_n^{-1}([r,1])$ is closed in X for all $n \in N$ and $r \in R$. Thus, $\bigcap_{n\in N} \Phi_n^{-1}([r,1])$ is closed in X for all $r \in R$. Fix $r_0 \in R$ and $\bigcap_{n\in N} \Phi_n^{-1}([r_0,1])$. Let $G_r = (\bigcap_{n\in N} \Phi_n^{-1}([r,1]))_X^C$. Then G_r is open in X for all $r \in R$ Assume, for the sake of contradiction, that no point of $\bigcap_{n\in N} \Phi_n^{-1}([r_0,1])$ belongs to every $\bigcap_{n\in N} \Phi_n^{-1}([r,1])$. Then $\bigcap_{n\in N} \Phi_n^{-1}([r_0,1]) \subseteq \bigcup_{r\in R} G_r$. Since $\bigcap_{n\in N} \Phi_n^{-1}([r_0,1])$ is a closed subset of X and X is compact, $\bigcap_{n\in N} \Phi_n^{-1}([r_0,1])$ is compact. Hence, there exist $r_1, \dots, r_k \in R$ such that $\bigcap_{n\in N} \Phi_n^{-1}([r_0,1]) \subseteq G_{r_1} \cup \dots \cup G_{r_k}$. Then $(\bigcap_{n\in N} \Phi_n^{-1}([r_0,1])) \cap \left(G_{r_1} \cup \dots \cup G_{r_k}\right)^C = (\bigcap_{n\in N} \Phi_n^{-1}([r_0,1])) \cap (\bigcap_{n\in N} \Phi_n^{-1}([r_1,1])) \cap \dots \cap (\bigcap_{n\in N} \Phi_n^{-1}([r_k,1])) = \emptyset$. Let $m = \max\{r_0, \dots, r_k\}$. Then $m \in R$. We have $(\bigcap_{n\in N} \Phi_n^{-1}([r_0,1])) \cap (\bigcap_{n\in N} \Phi_n^{-1}([r_1,1])) \cap \dots \cap (\bigcap_{n\in N} \Phi_n^{-1}([r_k,1])) = \bigcap_{n\in N} \Phi_n^{-1}([m,1]) \neq \emptyset$, because $m \in R$. This is a contradiction. Hence, there must exist $x \in \bigcap_{n\in N} \Phi_n^{-1}([r_0,1])$ such that $x \in \bigcap_{n\in N} \Phi_n^{-1}([r,1])$ for all $r \in R$. Therefore, $\bigcap_{r\in R}(\bigcap_{n\in N} \Phi_n^{-1}([r,1])) \neq \emptyset$. By Lemma 1,

$\bigcap_{r\in R}(\bigcap_{n\in N}\Phi_n^{-1}([r,1])) \subseteq \{x \in X: \forall y \in X, \min_{n\in N}\Phi_n(y) \leq \min_{n\in N}\Phi_n(x)\}$. Hence, $\{x \in X: \forall y \in X, \min_{n\in N}\Phi_n(y) \leq \min_{n\in N}\Phi_n(x)\} \neq \emptyset$. ■

Therefore, if X is compact and Φ_n is continuous on X for all $n \in N$, the set of points satisfying the MLS criterion is nonempty. We might therefore expect a group of players using some variety of aggregation rule following the MLS criterion to collectively select a point from this set. A fuzzy aggregation rule, or FAR, as exposited by Dutta (222) takes a set of fuzzy binary relations on X and returns a single fuzzy binary relation on X. In a sense, multiple individuals are coalesced into one. Note that our lemma and theorem relating to the MLS criterion does not require $|N| > 1$. Hence they apply to the case where $|N| = 1$.

Let us take a look at another criterion. Given a set of individuals each having a corresponding type-2 fuzzy maximal set, we might wish to select an alternative such that the mean membership in the type-2 fuzzy maximal sets is maximized. That is, we might wish to choose an alternative from the set $\left\{x \in X: \forall y \in X, \frac{\sum_N \Phi_n(y)}{|N|} \leq \frac{\sum_N \Phi_n(x)}{|N|}\right\}$. As with the MLS criterion, a theorem exists assuring us of the nonemptiness of this set under certain conditions.

Theorem 2: Let $N = \{1, \dots, k\}$ be a set of players with corresponding type-2 fuzzy maximal sets Φ_n. Suppose that Φ_n is continuous on X for all $n \in N$. Suppose further that X is compact. Then

$$\left\{x \in X: \forall y \in X, \frac{\sum_{n=1}^{k}\Phi_n(y)}{k} \leq \frac{\sum_{n=1}^{k}\Phi_n(x)}{k}\right\} \neq \emptyset$$

Proof: Since Φ_n is continuous on X for all $n \in N$, $\frac{\sum_{n=1}^{k}\Phi_n}{k}$ is continuous on X. Then since X is compact, $\frac{\sum_{n=1}^{k}\Phi_n}{k}$ attains a maximum on X. Therefore, there exists $x \in X$ such that for all $y \in X$, $\frac{\sum_{n=1}^{k}\Phi_n(y)}{k} \leq \frac{\sum_{n=1}^{k}\Phi_n(x)}{k}$. Hence, $\left\{x \in X: \forall y \in X, \frac{\sum_{n=1}^{k}\Phi_n(y)}{k} \leq \frac{\sum_{n=1}^{k}\Phi_n(x)}{k}\right\} \neq \emptyset$. ■

Note: A similar proof works to show that under the same conditions, the set $\left\{x \in X : \forall y \in X, \frac{\sum_{n=1}^{k} \Phi_n(x)}{k} \leq \frac{\sum_{n=1}^{k} \Phi_n(y)}{k}\right\}$ is nonempty.

Conclusion

There are many other criteria other than the two given that can be used, and each can be expected to relate to the topology of the type-2 fuzzy maximal sets in some way. This paper has shown how a useful fuzzy maximal set definition can be made for the fuzzy weak preference relation (preference function) using topology and integral calculus. Next steps include testing the practical applicability of such an approach and the quality of its results, as well as drawing more conclusions from the structure of the type-2 fuzzy maximal set.

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Remittances

Jacob Huju
Department of History
Creighton University
Omaha, Nebraska 68178
email: jacobhuju@creighton.edu

Abstract:

We present methods for measuring the desirability and availability to send remittances from the United States to other nations of the Western Hemisphere. We develop the metrics by employing six approaches: the analytic hierarchy approach, the Guiasu approach, the product approach, Yen's method, the set-valued statistical methods, and the precedence relation method. Our methods rely on expert opinion. The models weigh the relative importance of the three components of the overarching goal of having a good atmosphere for the sending of remittances.

Keywords:

remittances; Latin America; analytic hierarchy process; Guiasu method; product approach; Yen's method; set-valued statistical method; precedence relation method.

1. Introduction

G : Remittances - As defined by the Oxford English Dictionary, remittances are, "A sum of money or (formerly) a quantity of an item transferred from place or person to another." in the context of immigration, remittance (usually) means money sent to a home country from an immigrant living in the host country. While much of the literature on the subject discusses the flow of remittances in quantifiable terms, there is a lack of fuzzy mathematics on the subject. For this reason, this study will focus on the factors that affect the ability and desire to send remittances to a home country. Because the largest number of remittances and most information available involves Latin America and the Caribbean and the United States, this paper will highlight these movements of people and money. the three main factors ($G1, G2$, and $G3$) are broad groupings (Political, Social, Economic) with the more specific and mathematically calculable sub factors affecting their importance.

$G1$: Political - The political environment of the home country and the receiving country must be stable in order to allow an increase in transfer of remittances. Both countries must present forms of domestic and foreign trustworthiness to allow movement (of people or funds) across borders. The sub-factors under the political heading include immigration policy, corruption or lack of trust, and infrastructure.

$G2$: Social - Along with political stability, immigrants must be able to perceive a certain amount of social offerings, such as health care, education,. and land ownership, in order to desire to send remittances. The availability of these social entities in both the home and receiving countries will affect the sending of remittances, because without attaining a certain amount of education or general health, the immigrant would not be able to gain enough money in order to send a portion of it back to the home country.

$G3$: Economic - Many of the sub-factors under the heading of "economic" factors affect the social and political factors as well. Hoover, it deserves to be its own category in that, yet again, the factors of both the home country and the receiving country will affect the reception and sending of remittances. General macroeconomic stability is an important sub-factor as is the potentiality of market failures due to loss or economic "shock." Another important sub-factor is the environment of trade between the home and receiving countries.

$G11$: Immigration Policy ("red tape") - The ease of entry, re-entry, and re-re-entry will affect the availability of remittances. By studying the immigration policy of each country, we will be able to assign an "ease of entry" normalized value to each country.

$G12$: Corruption or Lack of Trust - this will clearly affect the ability and desire to send remittances, because if political graft is a portion of the money sent to the home country, then immigrants will have to send more to have the same effect. I will use the Transparency International Corruption Perceptions Index (CPI) to measure the corruption of each country.

$G13$: Infrastructure - This includes the availability of the movement of goods and people (i.e. road construction) and the incidence of international airports or seaports. This will also greatly affect social heading.

$G21$: Education - Using UN Millennium Development Goals (MDG) and the UN Development Programme (UNDP) data on education attainment and attendance, I will be able to quantify the availability of education in each country.

$G22$: Health Care - By analyzing the UNPD data on maternal health, disease presence, infant mortality, I will be able to assign a number for general quality of health care in each country. While this factor is hard to quantify, due to lack of records in some countries, I will have to make due with what is available.

$G23$: Property Rights - The World Bank has data on the instance of home ownership for different countries. This will affect the desire to send remittances because of the importance of "providing" for one's own. In addition, if immigrants can own property, then they have attained a level of economic prosperity in their receiving country.

$G31$: Economic Freedom - I plan to use the paper titled "Economic Freedom of 171 Counties in Year 2009: A weighted Average Approach;" in order to quantify the ability of immigrants to enter into business transactions in the home and receiving countries.

$G32$: Market Failure - The biggest indicator under this sub-factor will be unemployment. without jobs, immigrants cannot earn enough money to send remittances back. In addition, under this sub-factor will be the propensity toward economic "shocks" due to bank failures.

$G33$: Trade - This will be one of the hardest sub-factors to quantify because so many other factors affect it. Export-import balance, monetary policy, exchange rate stability, and other dynamics change the flow of trade and its percentage GDP within a country. There are many data available from the

World Bank and other organizations that will assist in quantifying this value.

$G34$: Income - Measured (and weighted from the GINI coefficient) as real GDP per capita, this will provide a small picture of the economic development of a country, which, I argue, will affect the availability of remittances and desire to send remittances back.

In conclusion, remittances are a very important part of development of many countries in Latin Americas and the Caribbean. For this very reason, this paper will study the eight different factors have on the desire and availability to send remittances from the United States to other nations of the Western Hemisphere with the goal of presenting a suggestion for further study and policy formation regarding remittances.

In this paper, we present a methods for measuring the desirability and availability to send remittances from the United States to other nations of the Western Hemisphere. We develop the metrics by employing six approaches: the analytic hierarchy approach, the Guiasu approach, the product approach, Yen's method, the set-valued statistical method, and the precedence relation method. Our methods rely on expert opinion. The models weigh the relative importance of the three components $G1, G2, G3$ of the overarching goal of having a good atmosphere for the sending of remittances.

2. The Linear Equations

We consider the weights of the experts in order to obtain the linear equations.

	$E1$	$E2$	$E3$	$E4$	Row Avg
$G1$	3	5	6	8	$5.50 = w_1$
$G2$	4	6	8	5	$5.75 = w_2$
$G3$	10	8	9	5	$8.00 = w_3$
Total	17	19	23	18	19.25

Analytic Hierarchy Process

We first consider the analytic hierarchy process. We take the row averages of the above matrix to determine the 3×3 reciprocal matrix R of the analytic hierarchy process, where the ij-th entry of R is w_i/w_j. We normalize the columns of R to obtain a matrix we call N. It is known that each column of N will be the same, []. Hence it suffices to normalize the first column of R. We have $\sum_{i=1}^{3} w_i = 19.25$. Thus the $i1$ entry of N is the transpose of the vector $(.286, .299, .416)$. Thus the row averages of the columns of n are given by the transpose of this same vector. Hence the weights of the Gj are given by this vector. We thus have

$$G = .286G1 + .299G2 + .416G3.$$

Guiasu Method

We next normalize the columns of the above matrix to obtain the following matrix. The row averages of the following matrix yield the weights of the Gj by the Guiasu method since the basic probability assignments are probabilistic, [].

Cols. Normalized	$E1$	$E2$	$E3$	$E3$	Row Avg	Row Product
$G1$	.176	.263	.261	.444	.286	.0053640
$G2$	.235	.316	.348	.278	.294	.0071841
$G3$	.588	.421	.391	.278	.420	.0269079
Total						.0394560

We thus have

$$G = .286G1 + .294G2 + .420G3.$$

Dempster-Shafer method

The theoretical underpinnings of this method can be found in []. Normalizing the Row Product column of the previous matrix yields the following equation for G.

$$G = .136G1 + .182G2 + .682G3.$$

Yen's Method

In [], Yen developed an approach that addressed the issue of managing imprecise and vague information in evidential reasoning by combining Dempster-Shafer theory with fuzzy set theory. In this section, we apply Yen's method to arrive at the degree of belief of certain subsets of a set $X = \{x_1, ..., x_k\}$. We assume we have n subsets of X, $A_j, j = 1, ..., n$, which are focal elements of a function m of the power set of X into the closed interval $[0, 1]$, i.e., $m(A_j) > 0, j = 1, ..., n$.

In the expert's table of weights, we divide each element of the column by the column's maximal entry. We thus obtain the following matrix from which we derive a linear equation for G by Yen's method.

	$E1$	$E2$	$E3$	$E4$	Row Avg
$G1$	.3	.625	.667	1	.648
$G2$	.4	.75	.889	.665	.666
$G3$	1	1	1	.625	.906
Total					2.220

It is shown in [] that if $m(A_j) = \frac{1}{n}$ for $j = 1, ..., n$, then $\text{Bel}(1_{\{x_i\}})$ is the average of the entries in the i-th row. Dividing each entry of the Row Avg column by 2.22, we thus obtain $\text{Bel}(1_{\{G1\}}) = .292$,$\text{Bel}(1_{\{G2\}}) = .300$,$\text{Bel}(1_{\{G3\}}) = .408$, where here $X = \{G1, G2, G3\}$ and $n = 4$. Hence

$$G = .292G1 + .300G2 + .408G3.$$

Comparison of Methods

We first compare Yen's method with the Analytic Hierarchy Process. Let $A = [a_{ij}]$ denote the original, $m \times n$ matrix of weights given by the n experts to the m goals. It is shown in [] that the AHP coefficients, A_i, of the linear equation are given by the equation

$$A_i = \frac{R_i}{\sum_{j=1}^{n} C_j}, i = 1, ..., m,$$

where $R_i = \sum_{j=1}^{n} a_{ij}, i = 1, ..., m$, and $C_j = \sum_{i=1}^{m} a_{ij}, j = 1, ..., n$.

Let $M_i = \vee\{a_{ij} | j = 1, ..., n\}, i = 1, ..., m$. Let Y denote the $m \times n$ matrix $[a_{ij}/M_i]$. It is shown in [] that the coefficients, $Y_i (i = 1, ..., m)$ determined by Yen's method are determined by normalizing the column consisting of the row averages of Y. That is,

$$Y_i = \frac{\frac{1}{n}\sum_{j=1}^{n} \frac{a_{ij}}{M_i}}{\sum_{i=1}^{m} \frac{1}{n}\sum_{j=1}^{n} \frac{a_{ij}}{M_i}}, i = 1, ..., m.$$

Thus

$$Y_i = \frac{\frac{R_i}{M_i}}{\sum_{i=1}^{m} \frac{R_i}{M_i}}, i = 1, ..., m.$$

It follows immediately that if $M_i = M_k$ for $i, k = 1, ..., m$, then $Y_i = A_i, i = 1, ..., m$ since $\sum_{j=1}^{n} C_j = \sum_{i=1}^{m} R_i$. This if we start with the maximum value of each column of A equal to 1, then Yen's method and the AHP yield the same coefficients.

A necessary and sufficient condition is given in [] for the Guiasu and the AHP to yield the same coefficients.

Set-Valued Statistical Method

In this section, we use the set-valued statistical method, [] to determine the coefficients of the linear equation expressing G in terms of $G2, G2$, and $G3$. We first give a brief description the method. Let $\mathcal{G}$ denote a finite set. In our case, $\mathcal{G} = \{G1, ..., Gm\}$ is a set of goals making up the overarching goal. Let $\mathcal{E}$ denote a set of subjects. In our case, $\mathcal{E} = \{E1, ..., En\}$ is the set of experts. The problem is to find the degree of membership $\widetilde{A}$ of $G1, ..., Gm$ in the set of factors (subgoals) making up the overarching goal, i.e., the coefficients of the linear equation in our application.

First choose an integer $q, 1 \leq q \leq m$, and then an Ej in order to carry out the following steps:

(1) Select $r_1 = q$ elements from G such that they are the first group of elements best fit to $\widetilde{A}$ by Ej. This yields a subset

$$\mathcal{G}_1^{(j)} = \{G_{i_1}^{(j)}, G_{i_2}^{(j)}, ..., G_{i_q}^{(j)}\}$$

of $\mathcal{G}$.

(2) Select $r_2 = 2q$ elements from $\mathcal{G}$ (which includes the q elements already selected in step (1) in such a way that all $2q$ elements are considered better fit to $\widetilde{A}$ than other elements of $\mathcal{G}$ by Ej. this yields the following subset of $\mathcal{G}$:

$$\mathcal{G}_2^{(j)} = \{G_{i_1}^{(j)}, G_{i_2}^{(j)}, ..., G_{i_q}^{(j)}, G_{i_{q+1}}^{(j)}, ..., G_{i_{2q}}^{(j)}\}.$$

(3) Continuing this process, we construct the s-th subset $G_s^{(j)} = G_1^{(j)} \cup ... \cup G_{s-1}^{(j)} \cup \{G_{i_{(s-1)+1}}^{(j)}, ..., G_{i_{sq}}^{(j)}\}$.

This process is continued until we obtain a positive integer t such that $m = tq + v$, where $1 \leq v \leq q$.

We next calculate $m(Ei)$, the average frequency of Ei using the following formula

$$m(Ei) = \frac{1}{n(t+1)} \sum_{s=1}^{t+1} \sum_{j=1}^{n} \chi_{G_s^{(j)}(Ei)}, i = 1, ..., m,$$

where $\chi_{G_s^{(j)}}$ is the characteristic function of the set $G_s^{(j)}$ in $G, j = 1, ..., n$.

A generalization of this method using differing sizes of q at each step can be found in []. We presented the one above since we choose $q = 1$.

The following matrix is determined by the weighting of $G1, G2, G3$ by $E1-E4$ as given in the first section.

	$G1$	$G2$	$G3$
$E1$	1	2	3
$E2$	1	2	3
$E3$	1	2	3
$E4$	3	1.5	1.5

We have $n = 4$ and $t = 2$. Thus

$$m(G1) = \frac{1}{12}(6) = .5, m(G2) = \frac{1}{12}(7.5) = .625, m(G3) = \frac{1}{12}(10.5) = .875.$$

We normalize $m(Gi), i = 1, 2, 3$, to obtain

$$G = .25G1 + .31G2 + .44G3.$$

Precedence Relation Method

Before employing the precedence relation method, we must determine a preference relation matrix for $G1, G2, G3$.

Let $X = \{G1, G2, G3\}$. We use the following binary relations $\rho_k, k = 1, 2, 3, 4$, on X to give the four experts preferences, where here we also let Gi denote expert k's weight for Gi from the normalized column matrix. Define $\rho_k : X \times X \to [0, 1]$ by $\forall (Gi, Gj) \in X \times X$,

$$\rho_k(Gi, Gj) = \begin{cases} (Gi - Gj + .5) \wedge 1 \text{ if } Gi > Gj, \\ 1 - [(Gj - Gi + .5) \wedge 1 \text{ if } Gi < Gj, \end{cases}$$

where $k = 1, ..., 4$ and $i, j = 1, 2, 3$. Let $r_{ij}^k = \rho_k(Gi, Gj)$ and let $R_k = [r_{ij}^k], k = 1, ..., 4$. Then we have the following matrices R_k that represent the ρ_k.

R_1	$G1$	$G2$	$G3$	R_2	$G1$	$G2$	$G3$
$G1$	.50	.44	.09	$G1$	.50	.44	.34
$G2$	.56	.50	.15	$G2$	.56	.50	.40
$G3$	.91	.85	.50	$G3$	.66	.60	.50

R_3	$G1$	$G2$	$G3$	R_4	$G1$	$G2$	$G3$
$G1$	.50	.41	.37	$G1$	.50	.67	.67
$G2$	.59	.50	.46	$G2$	.33	.50	.50
$G3$	.63	.54	.50	$G3$	.33	.50	.50

$$a_{ij}^k = \begin{cases} 1 \text{ if } r_{ij}^k > .5, \\ 0 \text{ otherwise.} \end{cases}$$

$$r_{ij} = \begin{cases} \frac{1}{4}\sum_{k=1}^{4} a_{ij}^k \text{ if } i \neq j, \\ 0 \text{ otherwise.} \end{cases}$$

$$R = \begin{array}{cccc} & G1 & G2 & G3 \\ G1 & 0 & .25 & .25 \\ G2 & .75 & 0 & 0 \\ G3 & .75 & .75 & 0 \end{array}$$

We use the precedence relation method, [,p.114].

Here we use the matrix $R = [r_{ij}]$ for our precedence relation matrix. We use the cut method as follows: Take $t \in [0, 1]$. Let

$$R_t = [c_{ij}^{(t)}],$$

where

$$r_{ij}^{(t)} = \begin{cases} 1 \text{ if } r_{ij} \geq t, \\ 0 \text{ otherwise.} \end{cases}$$

When t decreases from 1 to 0, and t_1 is the threshold where for the first time all the entries except the diagonal one in the i_1-th row of R_{t_1} are 1, then G_{i_1} is taken as an element (possibly not unique) in the first batch of the precedence set. We then remove G_{i_1} and continue the process. We obtain

$$R_{.75} = \begin{array}{cccc} & G1 & G2 & G3 \\ G1 & 0 & 0 & 0 \\ G2 & 1 & 0 & 0 \\ G3 & 1 & 1 & 0 \end{array}.$$

Thus we remove $G3$. We have

$$R^{(1)} = \begin{array}{ccc} & G1 & G2 \\ G1 & 0 & .25 \\ G2 & .75 & 0 \end{array}.$$

Clearly we remove $G2$ and then $G1$. This yields the following precedence order

$$\begin{array}{ccc} G3 & G2 & G1 \\ 3 & 2 & 1 \end{array}.$$

Thus

$$G = .17G1 + .33G2 + .50G3.$$

3. Linear Equations for $G1, G2, G3$

In this section, we apply the above methods to determine equation of the Gi in terms of the Gij.

	$E1$	$E2$	$E3$	$E4$	Row Avg
$G11$	5	1	8	8	$5.50 = w_1$
$G12$	6	6	10	9	$7.75 = w_2$
$G13$	6	6	9	7	$7.00 = w_3$
Total	17	13	27	24	20.25

	$E1$	$E2$	$E3$	$E4$	Row Avg
$G21$	8	6	9	8	$7.75 = w_1$
$G22$	8	2	10	8	$7.00 = w_2$
$G23$	6	7	8	6	$6.75 = w_3$
Total	17	19	23	18	21.50

	$E1$	$E2$	$E3$	$E4$	Row Avg
$G31$	7	6	10	4	$6.75 = w_1$
$G32$	8	6	10	6	$7.50 = w_2$
$G33$	8	6	7	10	$7.75 = w_3$
$G34$	8	4	8	5	$6.25 = w_4$
Total	31	22	35	25	28.25

Cols. Normalized	$E1$	$E2$	$E3$	$E4$	Row Avg	Row Product
$G11$	.294	.077	.296	.333	.250	.0022313
$G12$	.353	.462	.370	.375	.390	.0079877
$G13$	.353	.462	.333	.292	.360	.0158578
Total						.0260768

Cols. Normalized	$E1$	$E2$	$E3$	$E4$	Row Avg	Row Product
$G21$	.364	.400	.333	.364	.365	.0176484
$G22$	.364	.133	.370	.364	.308	.0065201
$G23$	.273	.467	.296	.273	.327	.0103022
Total						.0344707

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Cols. Normalized	$E1$	$E2$	$E3$	$E4$	Row Avg	Row Product
$G31$	.226	.273	.286	.160	.236	.0028232
$G32$	.258	.273	.286	.240	.264	.0048345
$G33$	.258	.273	.200	.400	.283	.0056347
$G34$	.258	.182	.229	.200	.217	.0021505
Total						.0154429

AHP Equations

$$G1 = .272G11 + .383G12 + .346G13$$
$$G2 = .360G21 + .326G22 + .314G23$$
$$G3 = .239G31 + .365G32 + .274G33 + .221G34$$

Guiasu Equations

$$G1 = .250G11 + .390G12 + .360G13$$
$$G2 = .365G21 + .308G22 + .327G23$$
$$G3 = .236G31 + .264G32 + .283G33 + .217G34$$

Dempster-Shafer Equations

$$G1 = .086G11 + .306G12 + .608G13$$
$$G2 = .512G21 + .189G22 + .299G23$$
$$G3 = .183G31 + .313G32 + .365G33 + .139G34$$

In a similar manner, we obtain the following tables and equations for Yen's method:

	$E1$	$E2$	$E3$	$E4$	Row Avg
$G11$	.833	.167	.8	.889	.672
$G12$	1	1	1	1	1
$G13$	1	1	.9	.778	.920
Total					2.592

	$E1$	$E2$	$E3$	$E4$	Row Avg
$G21$	1	.857	.9	1	.939
$G22$	1	.286	1	1	.8215
$G23$	.75	1	.8	.75	.825
Total					2.5855

$$G1 = .259G11 + .386G12 + .355G13$$

$$G2 = .363G21 + .318G22 + .319G23$$

	$E1$	$E2$	$E3$	$E4$	Row Total
$G31$	.875	1	1	.4	.819
$G32$	1	1	1	.6	.900
$G33$	1	1	.7	1	.925
$G34$	1	.667	.8	.5	.742
Total					3.386

$$G3 = .242G31 + .266G32 + .273G33 + .219G34$$

We now consider the set-valued statistical method and the precedence relation method.

$G1$: Political Atmosphere

R_{11}	$G11$	$G12$	$G13$
$G11$	.50	.44	.44
$G12$	.56	.50	.50
$G13$	.56	.56	.50

R_{12}	$G11$	$G12$	$G13$
$G11$	.50	.12	.12
$G12$	.88	.50	.50
$G13$	.88	.50	.50

R_{13}	$G11$	$G12$	$G13$
$G11$	.50	.43	.46
$G12$	.57	.50	.54
$G13$	.54	.46	.50

R_{14}	$G11$	$G12$	$G13$
$G11$	.50	.46	.54
$G12$	.54	.50	.58
$G13$	.46	.42	.50

$G2$: Social Development

R_{21}	$G21$	$G22$	$G23$
$G21$	.50	.50	.59
$G22$	.50	.50	.59
$G23$	.41	.41	.50

R_{22}	$G21$	$G22$	$G23$
$G21$	.50	.77	.43
$G22$	.23	.50	.17
$G23$	.57	.83	.50

R_{23}	$G21$	$G22$	$G23$
$G21$	.50	.46	.54
$G22$	.54	.50	.57
$G23$	.46	.43	.50

R_{24}	$G21$	$G22$	$G23$
$G21$	.50	.50	.59
$G22$	.50	.50	.59
$G23$	.41	.41	.50

$G3$: Economic Development

R_{31}	$G31$	$G32$	$G33$	$G34$
$G31$	.50	.47	.47	.47
$G32$	.53	.50	.50	.50
$G33$	.53	.50	.50	.50
$G34$	.53	.50	.50	.50

R_{32}	$G31$	$G32$	$G33$	$G34$
$G31$	.50	.50	.50	.59
$G32$	.50	.50	.50	.59
$G33$	.50	.50	.50	.59
$G34$	.41	.41	.41	.50

R_{33}	$G31$	$G32$	$G33$	$G34$
$G31$	.50	.50	.59	.56
$G32$	.50	.50	.59	.56
$G33$	.41	.41	.50	.47
$G34$	.44	.44	.53	.50

R_{34}	$G31$	$G32$	$G33$	$G34$
$G31$	.50	.42	.26	.46
$G32$	.58	.50	.34	.54
$G33$	.74	.66	.50	.70
$G34$	.54	.46	.30	.50

We next consider preference relations for $G1, G2, G3$.

We first consider $G1$.

R^1	$G11$	$G12$	$G13$
$G11$	0	0	.5
$G12$	1	0	.5
$G13$	.5	0	0

By the set-valued statistical method, we obtain

	$G11$	$G12$	$G13$
$E1$	1	2.5	2.5
$E2$	1	2.5	2.5
$E3$	1	3	2
$E4$	2	3	1

Now $n = 4$ and $t = 2$ so

$$m(G11) = \frac{1}{12}(5), m(G12) = \frac{1}{12}(11), m(G13) = \frac{1}{12}(8).$$

Normalizing, we have

$$G1 = .21G11 + .46G12 + .33G13$$

By the precedence relation method, we obtain

$G11$	$G12$	$G13$
1.5	3	1.5

Normalizing, we have

$$G1 = .25G11 + .50G12 + .25G13$$

We now consider $G2$.

R^2	$G21$	$G22$	$G23$
$G21$	0	.25	.75
$G22$	.25	0	.75
$G23$	.25	.25	0

By the set-valued statistical method, we obtain

	$G21$	$G22$	$G23$
$E1$	2.5	2.5	1
$E2$	2	1	3
$E3$	2	3	1
$E4$	2.5	2.5	1

Now $n = 4$ and $t = 2$ so

$$m(G21) = \frac{1}{12}(9), m(G22) = \frac{1}{12}(9), m(G23) = \frac{1}{12}(6)$$

Normalizing, we have

$$G2 = .375G21 + .375G22 + .25G23.$$

By the precedence relation method, we obtain

$G21$	$G22$	$G23$
2	2	2

Normalizing, we have

$$G2 = .33G21 + .33G22 + .33G23.$$

We next consider $G3$.

R^3	$G31$	$G32$	$G33$	$G34$
$G31$	0	0	0	.5
$G32$	.5	0	.25	.5
$G33$	.5	.25	0	.5
$G34$	.5	0	.25	0

By the set-valued statistical method, we obtain

	$G31$	$G32$	$G33$	$G34$
$E1$	1	3	3	3
$E2$	3	3	3	1
$E3$	3.5	3.5	1	2
$E4$	1	3	4	2

Now $n = 4$ and $t = 3$ so

$$m(G31) = \frac{1}{16}(8.5), m(G32) = \frac{1}{16}(12.5), m(G33) = \frac{1}{16}(11), m(G34) = \frac{1}{16}(8).$$

Normalizing, we have

$$G3 = .21G31 + .31G32 + .28G33 + .20G34.$$

By the precedence relation method, we obtain

$G31$	$G32$	$G33$	$G34$
1.5	3.5	3.5	1.5

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Normalizing, we have

$$G3 = .15G31 + .35G32 + .35G33 + .15G34.$$

4. Running the Model

	Remittances/capita	G	$G1$	$G2$	$G3$
Guiasu method			0.286	0.294	0.420
Mexico	250.5142857	0.097269601	0.068050988	0.104362771	0.112200865
El Salvador	543.4285714	0.081345424	0.067615704	0.101274997	0.076744007
Honduras	403.4285714	0.076881046	0.076279905	0.100209051	0.060654362
Nicaragua	136.3500000	0.075840941	0.090020862	0.100967577	0.048596445
Guatemala	342.0000000	0.084530407	0.102509090	0.099868969	0.061550786
Dominican Republic	348.7000000	0.078690140	0.057470656	0.096234716	0.080858396
Columbia	110.9950000	0.090574802	0.092695082	0.099173964	0.083111578
Chile	0.47058824	0.123718622	0.119407476	0.103559262	0.140765859
Brazil	26.50463542	0.085731259	0.081176716	0.088494945	0.086898106
United States		0.205446629	0.244323520	0.105951946	0.248619595

	$G11$	$G12$	$G13$
Guiasu method	0.25	0.39	0.36
Mexico	0.036363636	0.083123426	0.073727618
El Salvador	0.090909091	0.085642317	0.031910911
Honduras	0.145454545	0.062972292	0.043908541
Nicaragua	0.163636363	0.062972292	0.068201605
Guatemala	0.181818181	0.085642317	0.065705669
Dominican Republic	0.072727273	0.075566751	0.027271679
Columbia	0.127272727	0.093198992	0.068136925
Chile	0.109090909	0.168765743	0.073100304
Brazil	0.054545455	0.093198992	0.086646516
United States	0.018181818	0.18816877	0.461390232

	$G21$	$G22$	$G23$
Guiasu method	0.365	0.308	0.327
Mexico	0.105768206	0.104459714	0.102702703
El Salvador	0.099797420	0.098003165	0.106006006
Honduras	0.100117283	0.095119719	0.105105105
Nicaragua	0.103529161	0.099596850	0.099399399
Guatemala	0.103209297	0.092264270	0.103303303
Dominican Republic	0.090308135	0.093521686	0.105405405
Columbia	0.096918648	0.100650843	0.100300300
Chile	0.100757010	0.107470685	0.103003003
Brazil	0.099690799	0.099677262	0.065465465
United States	0.099904041	0.109235807	0.109609609

	$G31$	$G32$	$G33$	$G34$
Guiasu method	0.236	0.264	0.283	0.217
Mexico	0.101221722	0.086859686	0.138949079	0.120087520
El Salvador	0.106718694	0.082405345	0.062272235	0.056130607
Honduras	0.089097413	0.062360802	0.056879740	0.032567533
Nicaragua	0.090103901	0.051224944	0.031887127	0.022048304
Guatemala	0.089407101	0.075723831	0.042031912	0.039468148
Dominican Republic	0.089329679	0.097995546	0.068895893	0.066397374
Columbia	0.095647327	0.126948775	0.040579139	0.071614912
Chile	0.122466360	0.126948775	0.191224269	0.111672137
Brazil	0.088555457	0.131403118	0.045651532	0.084742910
United States	0.127452347	0.158129176	0.321629075	0.395270555

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Finding the Flaws in Wykstra's Formulation of Skeptical Theism:

An Alternative Argument for Atheism

Megan Hall

Creighton University

At first glance the argument for skeptical theism as formulated by Stephen Wysktra seems to be acceptable reasoning in support of theism despite the problem of evil. On further examination, however, the argument only briefly sidesteps the greatest obstacle for theism. I find flaws with Wykstra's formulation including his use of the analogy between fleas in a garage and God's capacity and intention for good. He also assumes moral perfection of God, or at least moral superiority, which has not been successfully established by any of the traditional philosophical arguments for theism. Showing the weaknesses of skeptical theism has serious implications against choosing theism over atheism in the face of evil. This analysis eliminates skeptical theism as an acceptable argumentative strategy for navigating past the problem of evil. Given the evidential problem of evil, I believe that if a choice must be made between theism and atheism, the weaknesses of Wykstra's skeptical theism argument lead us to choose atheism.

Although agnosticism is currently a possible position in the discussion of theistic belief, I believe indefinite suspension of judgment is a poor approach. While someone may be hesitant to choose, this indecisiveness is unwarranted. Continual reevaluation of arguments and counterarguments as new evidence is presented is an epistemic responsibility and allows the option to change one's mind. Therefore, I assert that there is no danger in choosing between atheism and theism. I find it unnecessary to continue the indecision of agnosticism 'for the certainty of avoiding error,' as philosopher William Rowe writes in *Philosophy of Religion: An Introduction* (102). Ultimately, either atheism or theism is true; and thus, in opposition to Rowe's claim, I believe that the agnostic is always wrong, no matter the truth. Hence, I argue that it is more admirable to choose either atheism or theism than to wade ambivalently in agnosticism.

After examining the Cosmological, Ontological, and Design Arguments as the traditional philosophical arguments for a theistic God, Rowe judges that "taken singly or together, they fail to establish theistic belief" (Rowe 66). He continues to refer to the arguments as plausible for God's existence. Rowe offers the possibility of their significance for theism as he suggests they provide some rational support for the belief. However, these arguments do not prove God's existence as has previously been claimed (Rowe 67). This lack of argumentative support permits theism to crumble further when evil is identified in the world. I believe the existence of unnecessary evils, like the fawn's suffering, discredits theism and has not been successfully surpassed. Atheism is left as the more rational and reasonable option. Stephen Wykstra attempts to justify evil with skeptical theism by purporting the existence of divine goods that we are psychologically incapable of understanding. The following is the reconstruction of Wykstra's argument from a handout written by William Stephens:

1) If we look in a garage and see no dog, then we're entitled to believe that there's no dog in the garage.
2) BUT, if we look in a garage and see no fleas, then we're NOT entitled to believe that there are no fleas in the garage.
3) Similarly, our not being able to think of a good that might justify God in permitting the fawn's suffering does NOT entitle us to think there isn't such a good.
4) Therefore, the fact that we cannot even imagine what such a good would be is just what we should expect to be true of a God who knows goods that are inscrutable to us.

My first difficulty with Wykstra's argument arises from the analogy he draws from premise 2 in formulating premise 3. He is essentially comparing fleas and God's capacity and intentions for good. Regardless of the unflattering image of God that this offers, the analogy fails because of the dissimilarity between the two. With premises 1 and 2, I agree that a dog's presence should be easily perceived, but we cannot claim there are no fleas just because we fail to sense them. However, the presence of fleas in the garage can be objectively tested beyond our original sensory perception. For example, I might place a flea-less dog in the garage and check to see if he is infested with fleas after some time. If so, then I have confirmed the presence of fleas in the garage even though I did not initially see them. If the dog is not infested and no insecticides have been used on the dog or the garage during this time, then I can conclude that there were no fleas in the garage as I originally perceived. Wysktra attempts to create an analogy, but we cannot similarly test for the presence of God or for his divine concepts of good. He might further argue that this would be correct as it is precisely the intent of his conclusion: the divine goods are inscrutable to us and no human tests are capable of measuring for these goods. Yet I still find that the connection he is attempting to create is very weak. The two examples are far too dissimilar to function as a successful analogy. Wykstra would have to find something much closer to God in perceptibility, or lack thereof, which cannot be objectively tested as the fleas can. This task may prove to be quite difficult considering theism's assertion of God's uniqueness. This weakness in Wykstra's argument diminishes his ability to assert that divine and incomprehensible goods justify evil. We then return to atheism as our only valid option given the undeniable prevalence of evil in the world.

The other major complication I find in Wykstra's argument is his assumption of God's moral perfection, or at least moral superiority, in his conclusion. Even if I grant the analogy between premises 2 and 3 to be sufficient, it is still apparent that the conclusion does not necessarily follow from premise 3. Wykstra appears to be flirting with the fallacy of begging the question. He attempts to reassert theism's strength in the presence of evil with a conclusion that requires truth of theism, and consequently the theistic attributes. He contends that evil exists justifiably because God has moral perfection and supreme wisdom such that we should anticipate our inability to understand God's intentions of good in the face of evil. The theistic God must exist for us to have these expectations of him and ourselves. Without this assumption, the skeptical theism argument fails to accomplish the goal of justifying evil. If a morally perfect, omniscient creator of our world is not assumed, then we have no basis to accept Wykstra's answer that evil is justified by these superior expectations. The perfections implied by the theistic God transform evil into a problem that must be explained.

Wykstra might object to my argument by following William James in suggesting that the chance at receiving the vital goods is worth the risk of error (Rowe 102). However, I believe the potential for these vital goods provides a reason to suspect misguided intentions in human action. It seems more gratifying and awe-inspiring to witness and receive the true good spirit of humankind rather than to wonder if each kind act is used merely as means to earn the vital goods. This is not fatal to Wykstra's objection because some people may be able to overlook this pessimistic consideration and accept all good acts as admirable, regardless of the motivation.

However, I believe that Wykstra's derivation of inscrutable goods from theistic attributes renders humans morally paralyzed. We become incapable of judging our own actions, never mind anyone else's. By skeptical theism we can no longer assess the world by our conceptions of good and evil. How can we strive for moral perfection if it is constituted by a concept that we cannot comprehend? Our quest for moral perfection becomes aimless if we are unable to grasp the end we are working toward. Our notions become worthless if it can be argued that God has incomprehensible, yet superior goods that surpass our best efforts. We are forced to assume that any and all evils may have divine justification in order to maintain consistency. Why offer any moral concern to the beaten dog, the injured horse, the battered wife or the starving child if the fawn's suffering is justifiable by God? This position eliminates our capacity to distinguish when human intervention is warranted from when we interfere with God's goods.

I think that atheism offers liberation rather than moral handicap. For an atheist, evil is not a problem. There is no unattainable, incomprehensible example of a morally perfect creator to which every action and event is compared. Even without a standing example of moral perfection, we can function within our own ideals of morality, continually striving to be better and making judgments on moral evil, relative to our concepts of goodness. Natural evils like the fawn's suffering can be explained in conjunction with Darwin's theory of evolution by natural selection. Nature maintains relatively stable and sustainable populations, before human intervention, by eliminating the weak through predation, competition, disease, or natural disasters. Any death or suffering that occurs in the process is essentially for the betterment of nature as a whole and need not be explained beyond the continual cycle of life and death. Therefore, atheism establishes itself as a more rationally supported and reasonable position by eliminating evil as a problem.

Not only does Wykstra's Skeptical Theism argument fail to make a valid analogy between unseen fleas and God's inscrutable goods, it also assumes God's existence to account for evil and reaffirm belief in theism. I find that these weaknesses return us to analysis of the problem of evil. Atheism relinquishes us from the view of evil as a problem. Without the need to compare our world to a morally perfect being, we can reinterpret occurrences of "evil" as necessary consequences of life, including the case of the charred fawn. For an atheist, moral evils can still be justly analyzed by our conceptions of good and judged accordingly. The skeptical theist must instead acknowledge the infinite possibilities of divine goods which override our assessments and justify evils. Since I believe the only valid positions in the debate of God's existence are theism or atheism, agnostics must confront their hesitation and choose. Given Wykstra's inefficacy in formulating skeptical theism to explain the problem of evil, theism appears to be the weaker position. Thus, I propose that atheism is the only standing option.

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Creighton
UNIVERSITY